Oldest-Edge Deletion in a Preferential Attachment Graph

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Gniezno

Preferential Attachment Graphs

Random graph model popular for evolving real-world networks

Based on concept “the rich get richer”

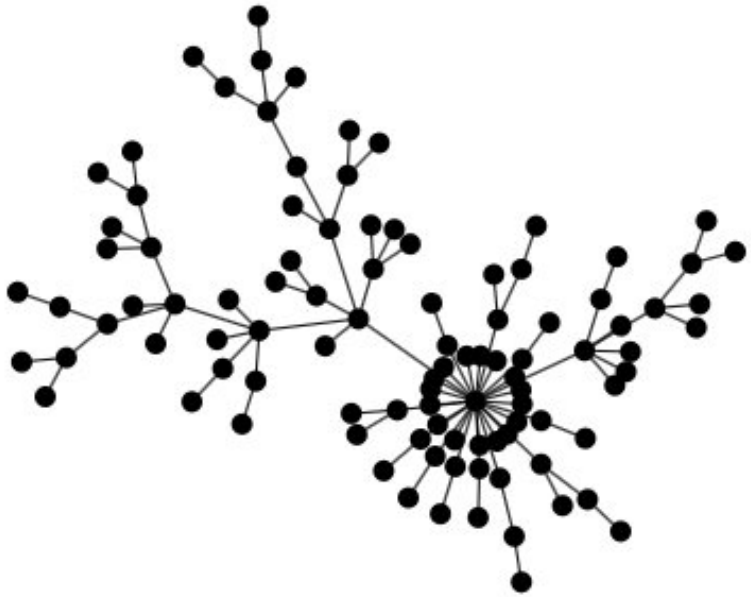
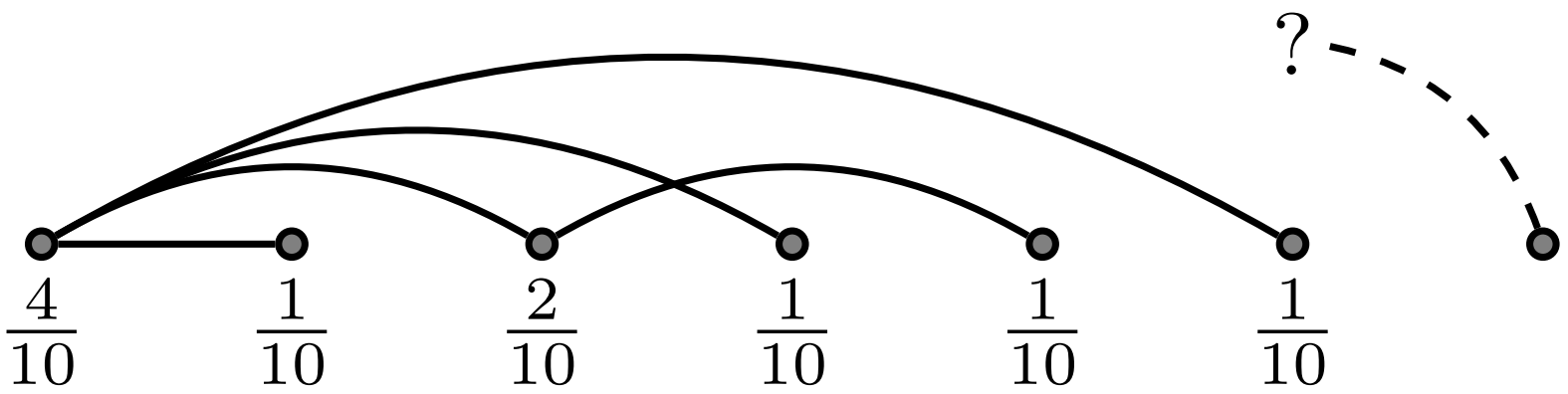


Figure: isomorphis.wordpress.com

Preferential Attachment Graphs

Barabási-Albert model: Given a graph G_t on $[t]$, form G_{t+1} by adding vertex $t+1$ along with $m \geq 1$ edges.

Probability of edge choosing v is proportional to degree of v in G_t.



Preferential Attachment Graphs

Barabási-Albert model popular since it follows a power law.

Theorem (Bollobás, Riordan, Spencer, Tusnády 2001)

Let X_k denote the number of vertices of degree k in G_n. There exists an $\alpha > 0$ such that for large k,

$$\lim_{n\to\infty} \frac{X_k}{n} = k^{-3}(\alpha + o_k(1)).$$

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Real-world networks often follow power laws.

Most random graph models, on the contrary, follow an exponential law:

$$\frac{X_k}{n} \longrightarrow c^k(\alpha + o_k(1)), \quad 0 < c < 1$$

Edge Deletion

We consider a variation of the Barabási-Albert model with online oldest-edge deletion, to be defined later.

We will consider the following questions:

- What is the probability distribution of $\deg v$ for a fixed vertex v?
- What is the degree sequence of the graph?
- What is the component structure of the graph?

Alternative View ($m = 1$)

Suppose an edge e is added to a graph G. Then

e chooses a vertex in G with probability proportional to degree

$\Longleftrightarrow$

e chooses an edge f uniformly at random from G, and chooses an endpoint of f uniformly at random

Indeed,

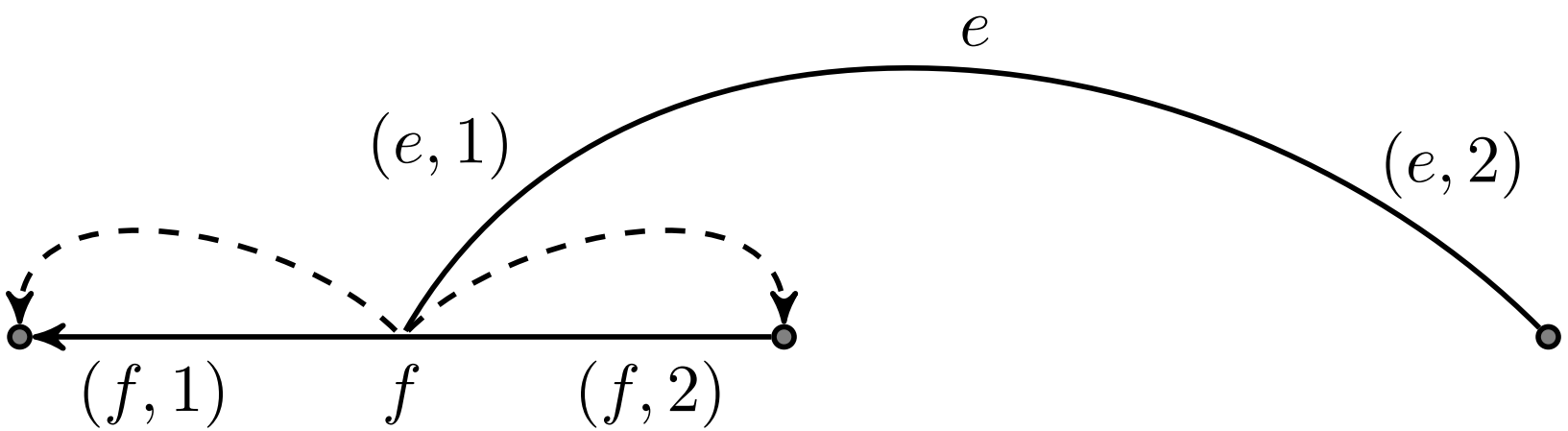
$$\frac{\deg v}{\sum_u \deg u} = \frac{1}{2|E(G)|} \sum_{f \in E(G)} [v \in f]$$

Alternative View ($m = 1$)

Edges are oriented from right to left. The base is fixed while the tip is random. If e chooses (f, j) we say that

- $j = 1$ corresponds to choosing the tip of f,
- $j = 2$ to choosing the base of f.

Then $(e, 1)$ and (f, j) are incident to the same vertex, while $(e, 2)$ is incident to the vertex e was added with



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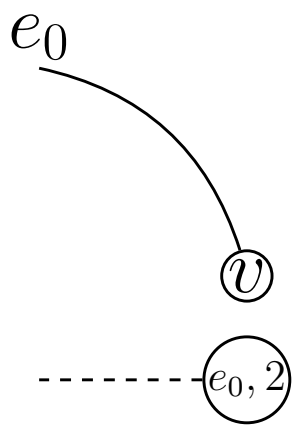
Edges choosing edges sets up partition of pairs (e, j)



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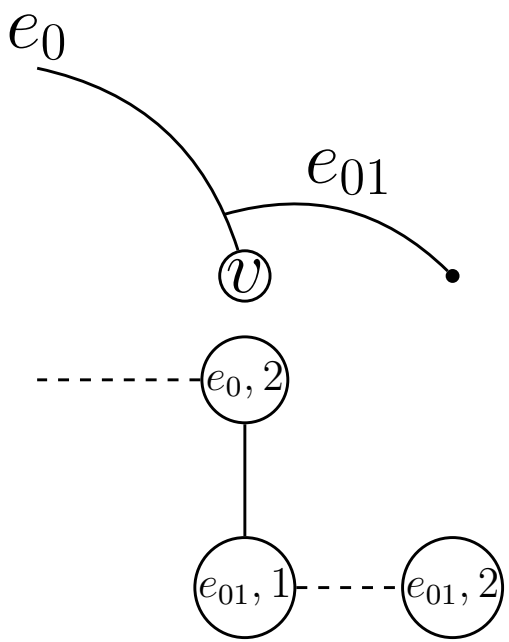

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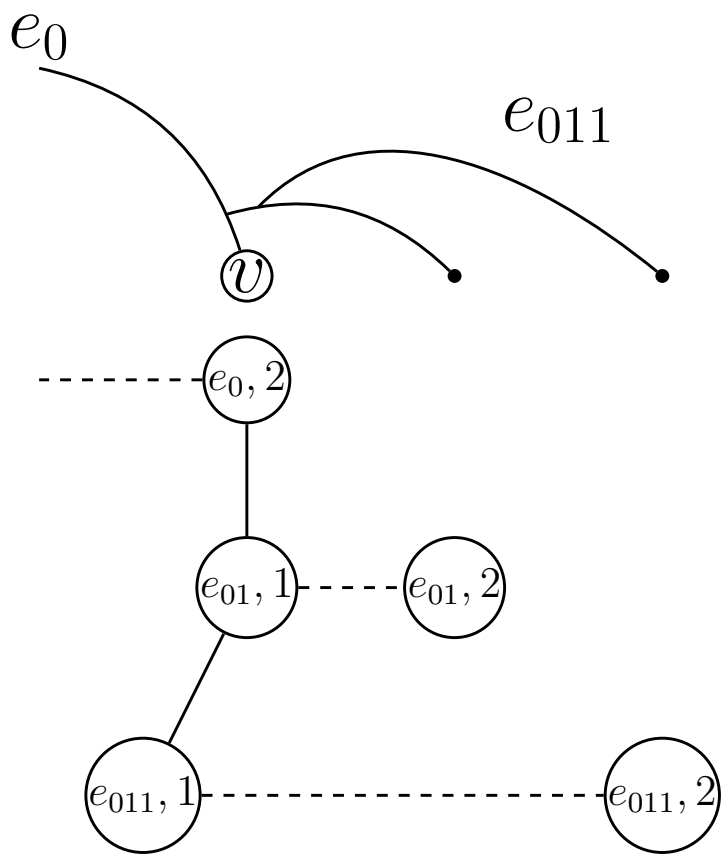

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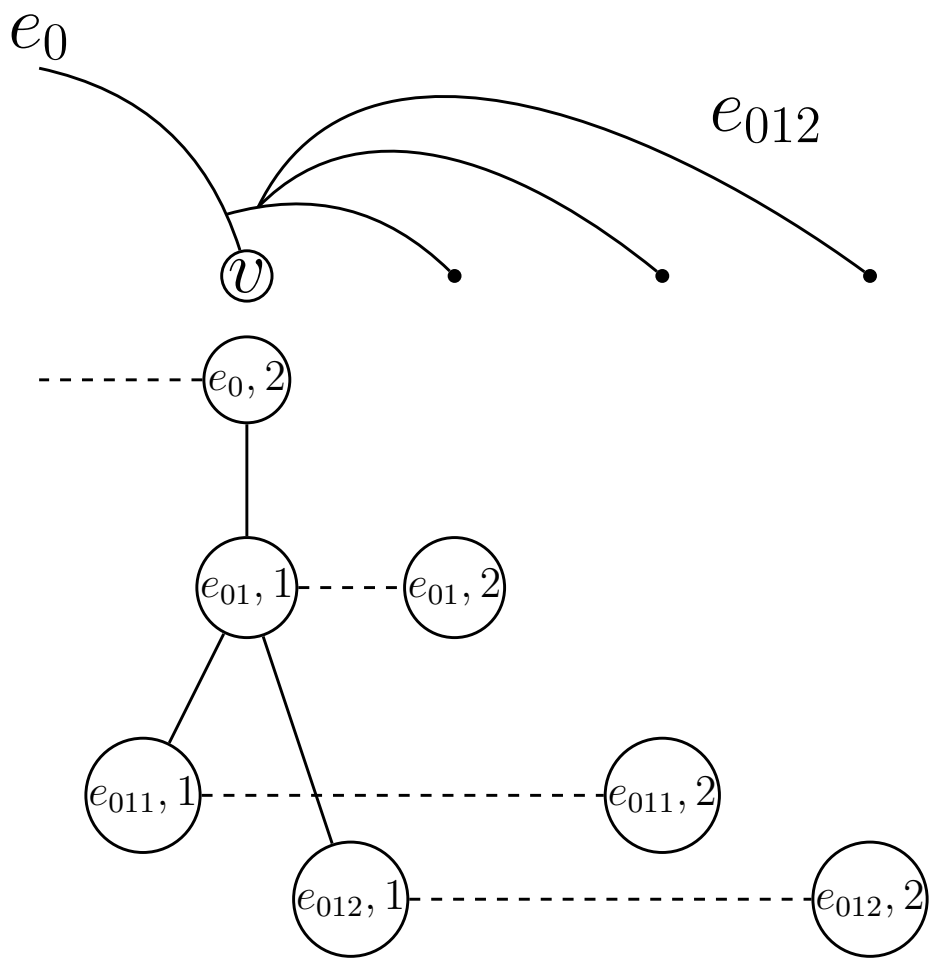
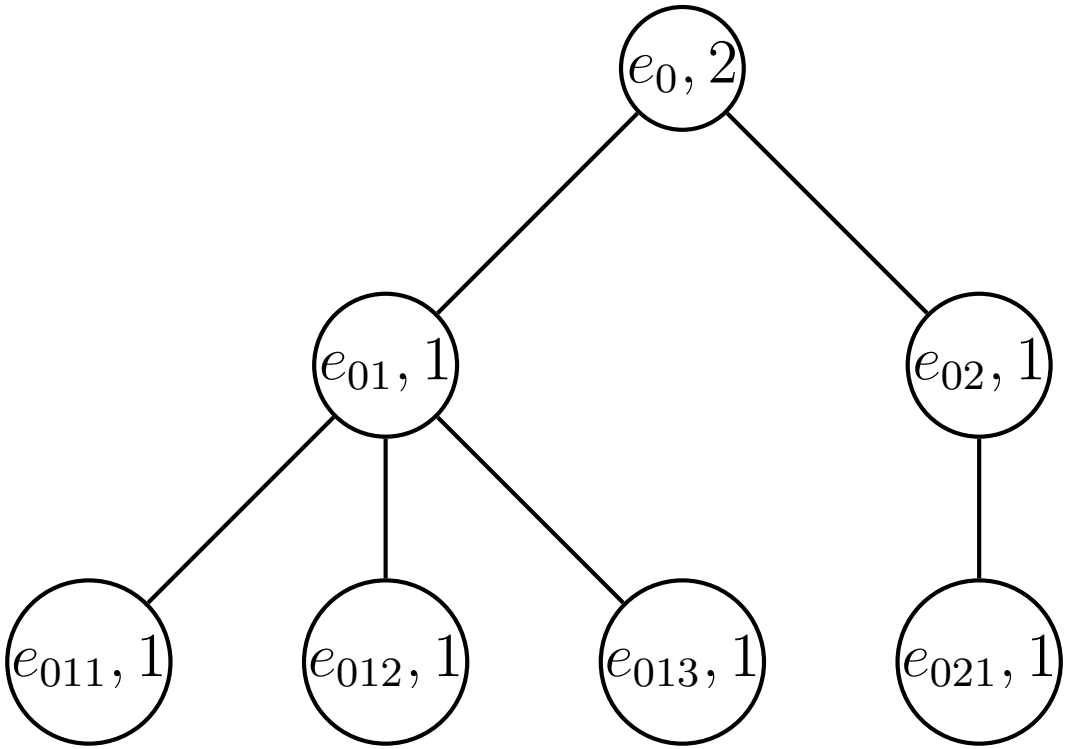


Alternative View ($m = 1$)

Degree of v equals size of tree $T(v)$



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Edge e is added at time t attaches to any (f, j) with probability $\frac{1}{t}$

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Size of tree is geometrically distributed, average depending on age of v

Edge Deletion

Introduce parameter $1/2 < p < 1$.

With probability p add vertex with m edges like before,
with probability $1 - p$ remove m oldest edges.

Vertex set $\{1, \ldots, \nu_t\}$ for some random ν_t.

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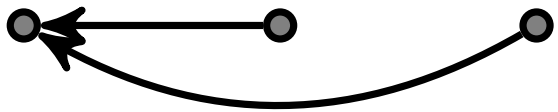

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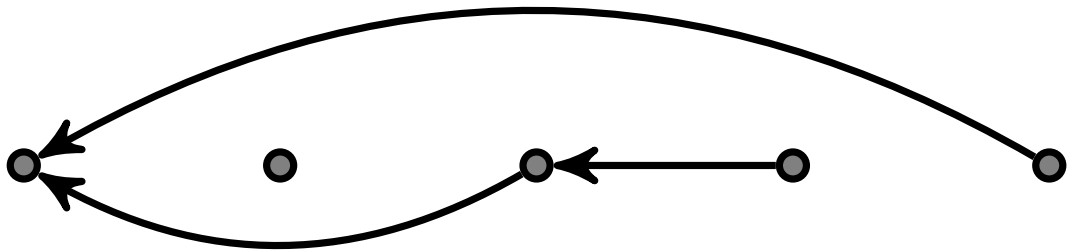

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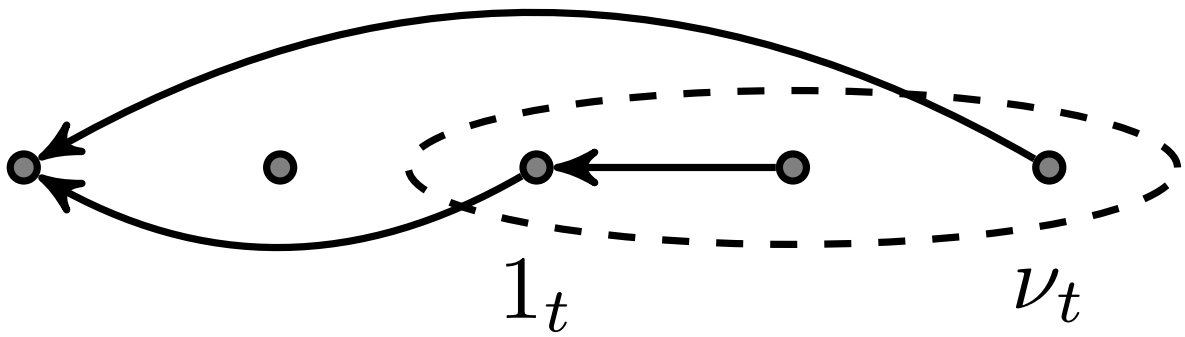

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Edge Deletion

Goals:

1. Determine degree sequence. Power law?
2. Is there a giant component?

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Tools:

1. Couple degree of a vertex to Crump-Mode-Jagers process (branching Poisson process)
2. Generate graph off-line

Edge Deletion

Suppose $m = 1$. We will consider $m > 1$ later.

Theorem (J.)

There exist $p_{n,v}$ $q_{n,v}$ such that for any v and $k \geq 0$,

$$Pr\{\deg v = k\} \approx \begin{cases} 1 - q_{n,v}, & k = 0, \\ q_{n,v} p_{n,v} (1 - p_{n,v})^{k-1}, & k > 0. \end{cases}$$

In other words, $\deg v$ is either zero or geometrically distributed.

The proof explicitly provides $p_{n,v}$, $q_{n,v}$.

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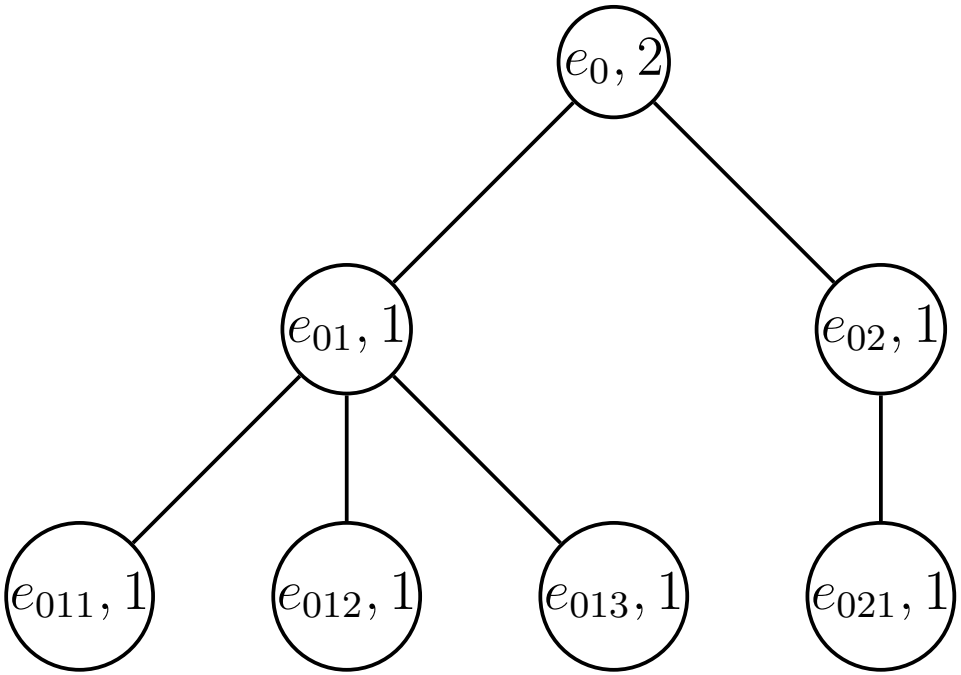
Theorem (J.)

There exists a $p_0 \approx 0.83$ such that if $p > p_0$ then G_n follows a power law whp, and if $p < p_0$ then G_n follows an exponential law whp.

Degree Distribution

Need to keep track of which edges are still in graph



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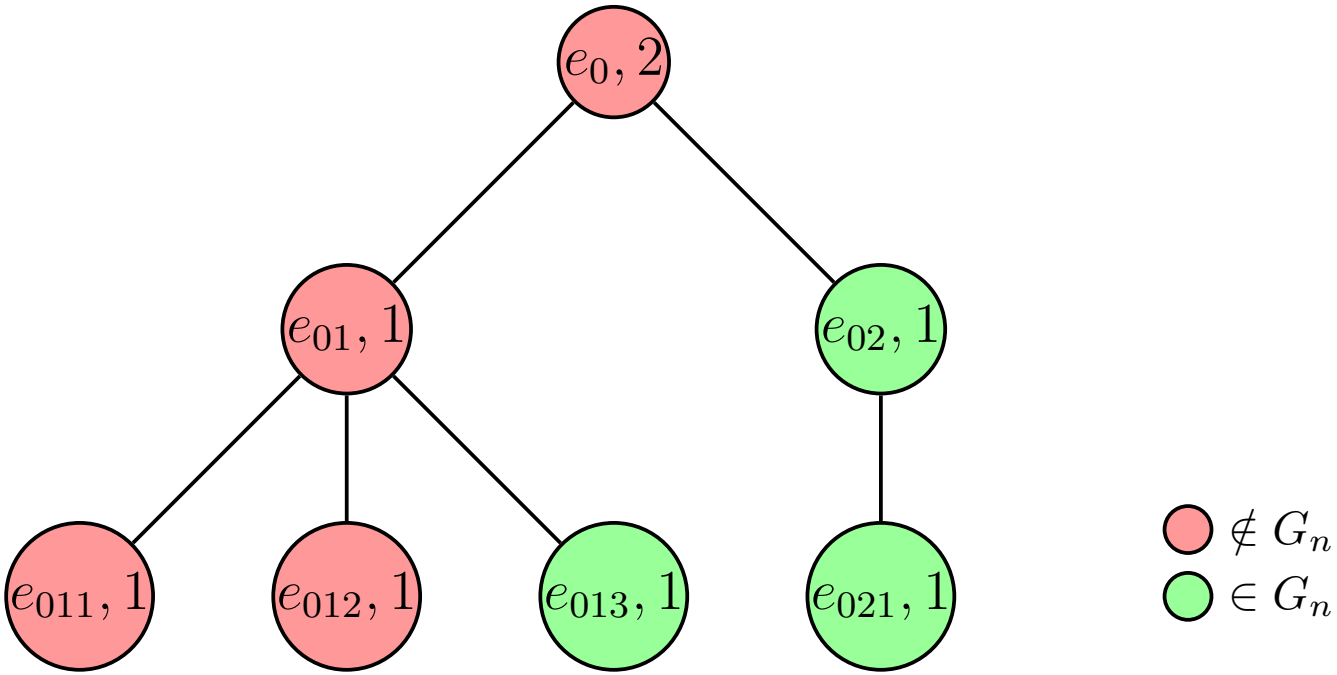

This vertex has degree 3 in G_n

Degree Distribution

The sequence $\sigma \in \{\pm 1\}^n$ of additions/removals is essentially a simple random walk

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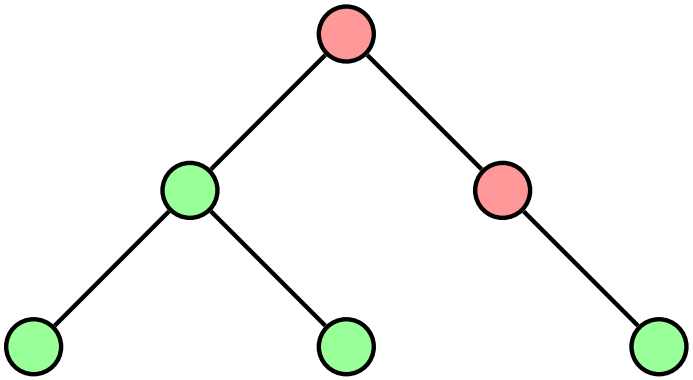
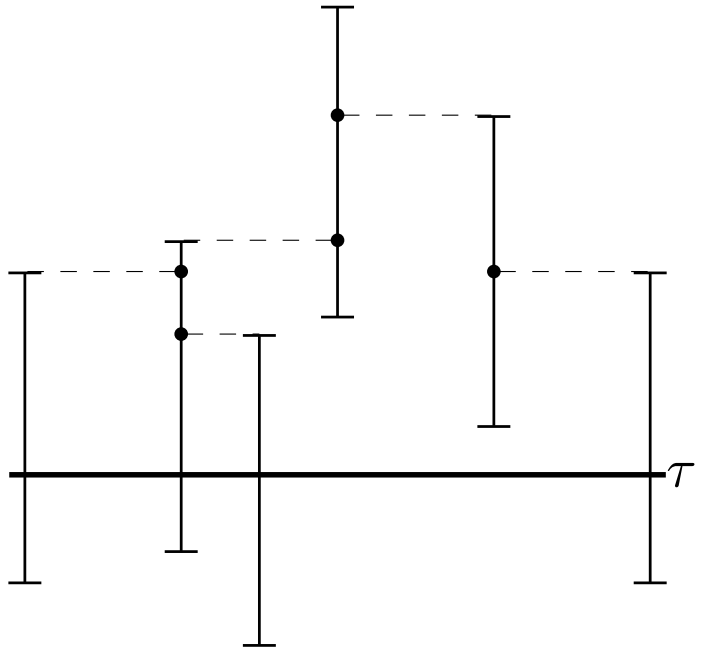
An edge e added at time t is removed at time $\approx \frac{p}{1-p} t$ whp

Determining existence of an edge is almost deterministic as function of its age!

Degree Distribution

Rescaling time logarithmically, $\tau = \log_{\frac{p}{1-p}} t$, an edge added at time τ is removed at time $\approx \tau + 1$





Branching Poisson processes now have life-time 1. Degree is number of active processes at time τ

Degree Distribution

Crump, Mode (1969): Number of active processes in this Crump-Mode-Jagers process at time τ has probability generating function $F(s,\tau)$ given by

$$F(s,\tau) = s \exp\left\{\alpha \int_0^\tau (F(s,u)-1)du\right\}, 0 \leq \tau \leq 1$$

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Solution is zero-or-geometric distribution, parameters depending on τ

Degree Sequence

G_n behaves differently depending on p.

Theorem (J.)

Let X_k denote the number of vertices of degree k in G_n. Then

$$\frac{X_k}{n} \sim \begin{cases} \alpha^{k(1+o_k(1))}, & \alpha < 1, \\ k^{-\eta-1}(1+o_k(1)), & \alpha > 1, \end{cases}$$

where $\eta = \eta(p) \in (2, \infty)$.

As a continous function of $p \in (1/2, 1)$, α increases from $1/2$ to ∞ with $\alpha = 1$ at $p \approx 0.83$.

Why a Phase Transition?

- When $\alpha < 1$, the CMJ process is ultimately extinct whp. Old vertices are isolated.

- When $\alpha > 1$, the process survives with some positive probability, and the old vertices have high expected degree.

Larger m

If $m > 1$, we can generate the $m = 1$ graph and merge groups of m adjacent nodes

Degrees are sums of m iid random variables

Degree sequence is essentially unaffected, in particular phase transition occurs at $p \approx 0.83$ independently of m

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Larger m lead to a giant component, as we will now see

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So far we have

- Found probability distribution of $\deg v$, and
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Remainder of talk:

- Show how to generate graph off-line,
- Find conditions for unique giant component

Giant Component

Theorem (J.)

If $m = 1$ the largest component of G_n has size $O(\Delta(G_n) \log n)$ whp. If $m > 1$ the largest component has size $\Omega(n)$ while all other components have size $O(\log n)$ whp.

Furthermore, if $\alpha > 1$ then whp the largest component contains all but a $(13/14)^m$ proportion of the edges, and the number of isolated vertices is $\xi n(1 + o_n(1))$ for some $0 < \xi < p$.

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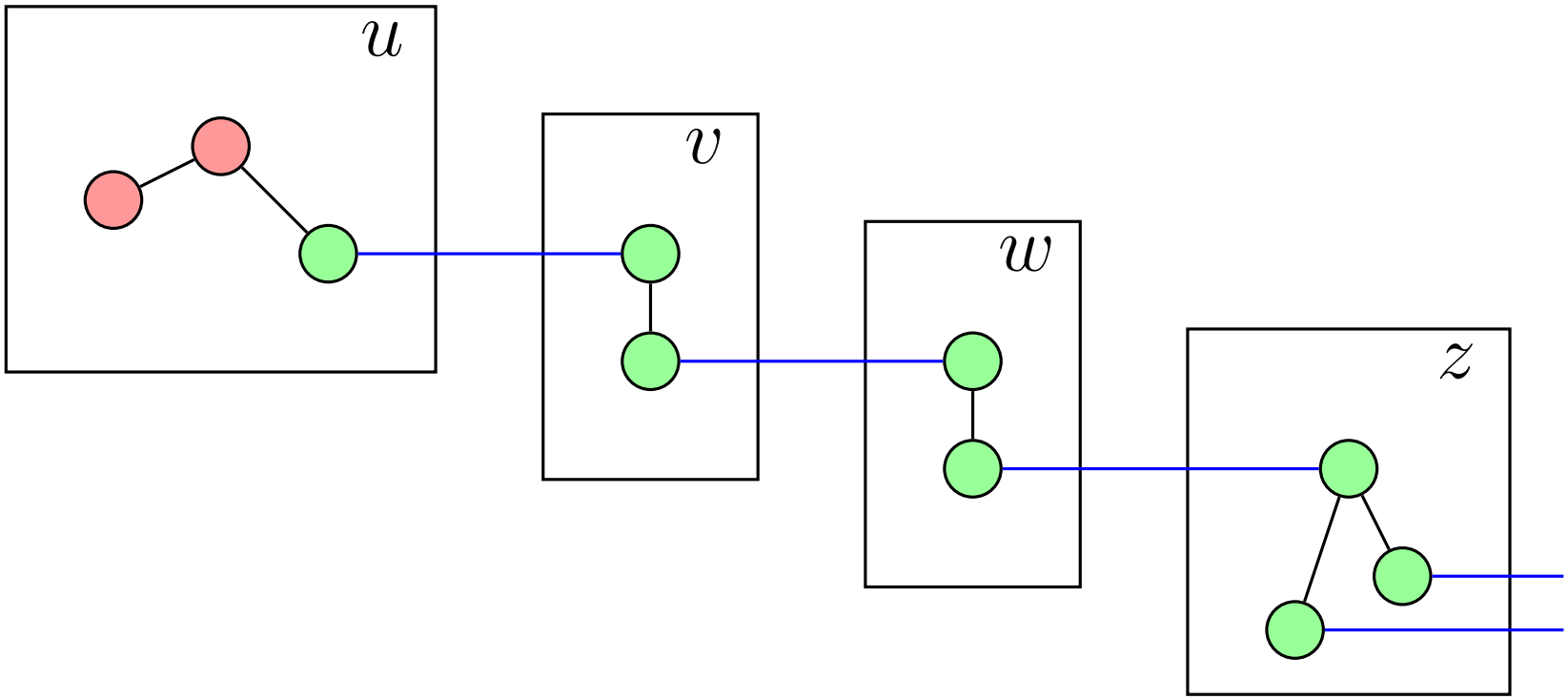
- With high probability maximum degree $\Delta(G_n)$ is $O(\log n)$ when $\alpha < 1$ and $O(n^{1/\eta}) = o(n^{1/2})$ when $\alpha > 1$.
- Existence of linear-sized component is independent of p.

Exploring the Graph

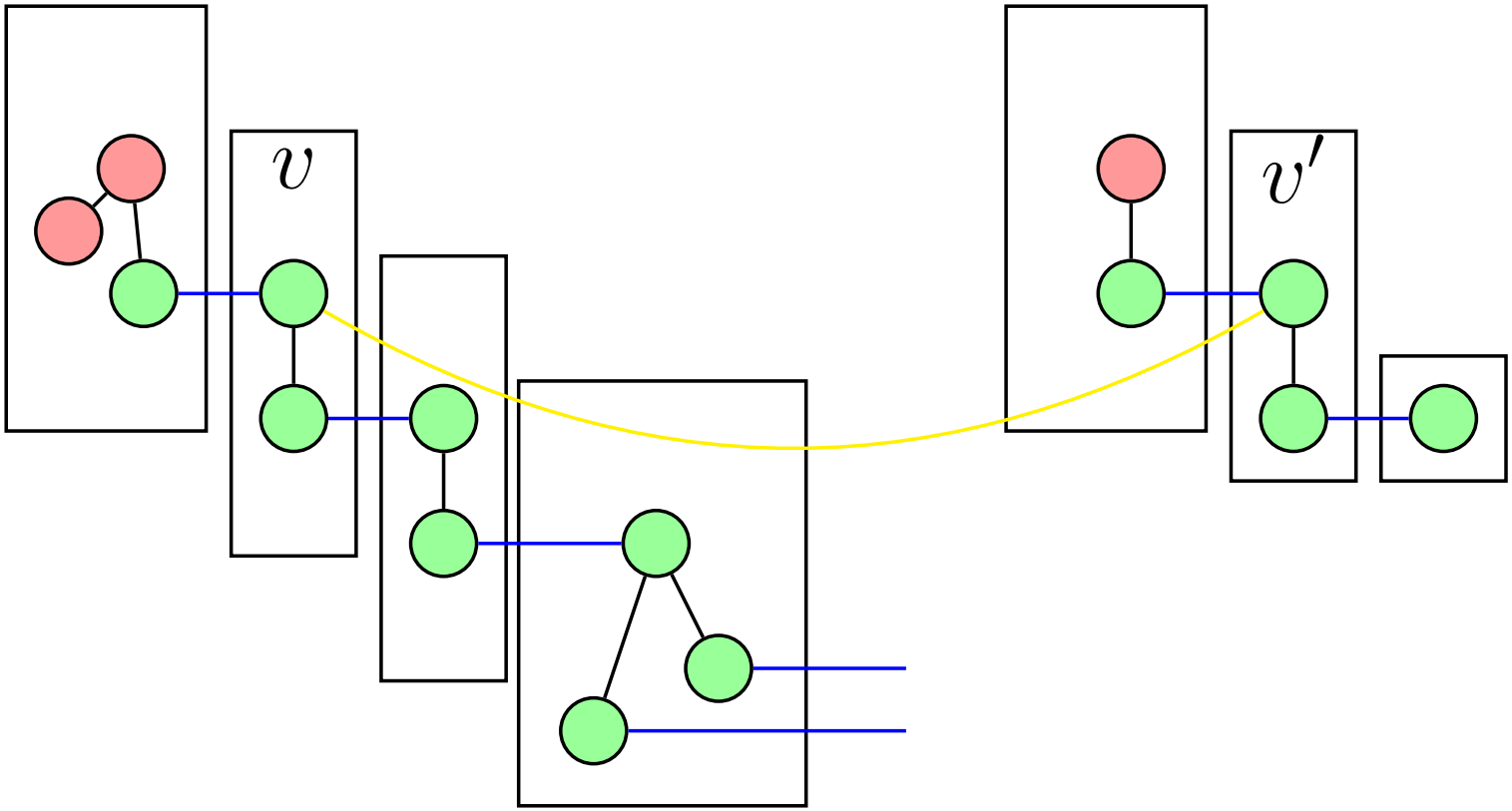
When $m = 1$, the component structure is as below



- Box — all half-edges incident to vertex
- Blue edge — connects two halves of a living edge

Exploring the Graph

When $m = 2$, we merge v with a vertex v' (yellow edge)



Larger m gives more yellow edges

Exploring the Graph

Starting with some edge e, the exploration process is:

1. Find all other edges incident to e, (black edges)
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This can be turned into a supercritical Galton-Watson process, and the giant component follows in a standard way

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To explore the tree $T(v)$ from any starting point, we need an offline generation scheme (need to go up the tree)

Conditioning on the sequence $\sigma \in \{\pm\}^n$ of additions/removals allows us to do this

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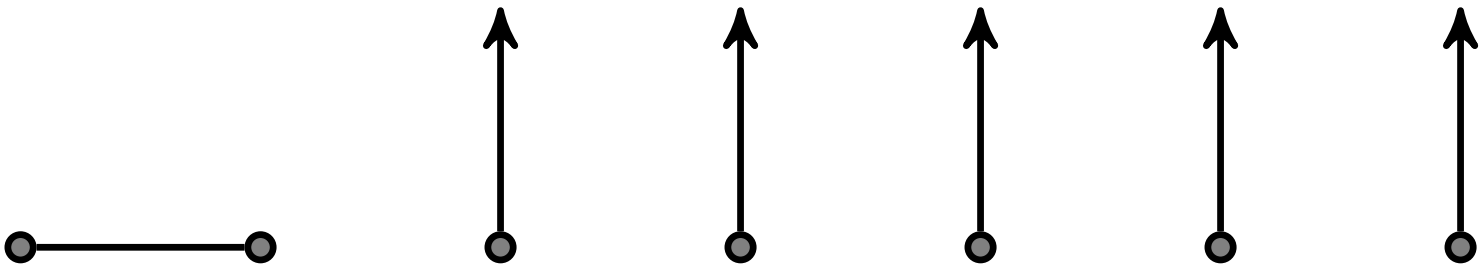

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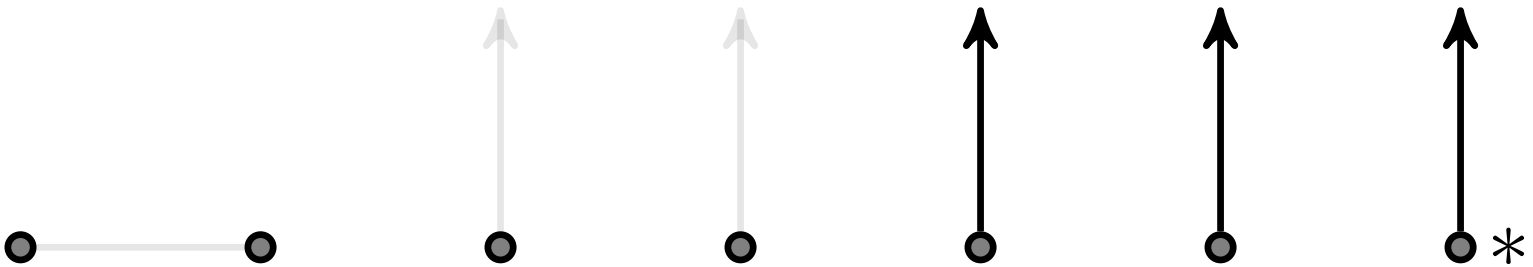

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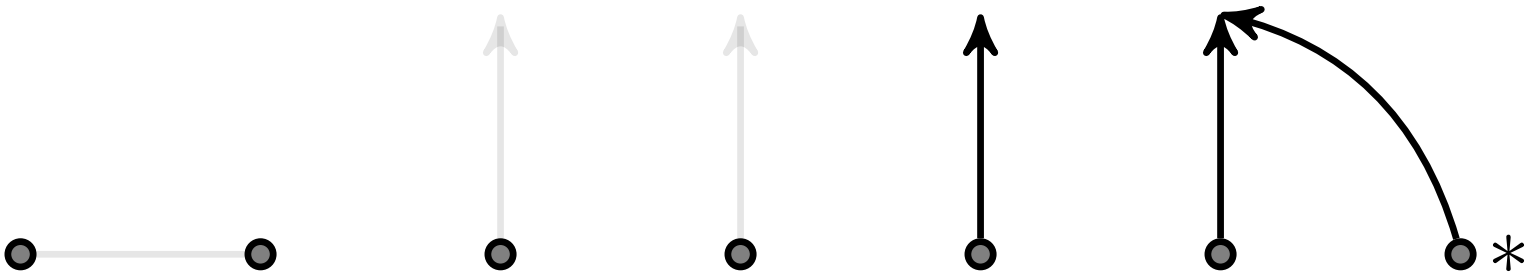

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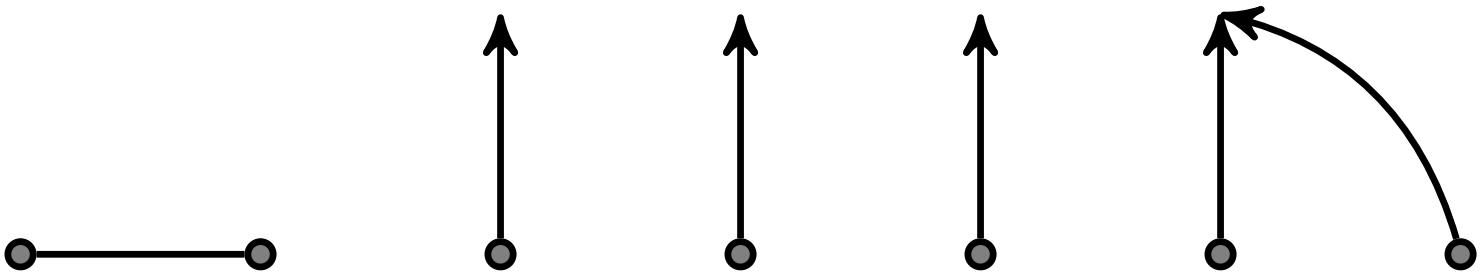

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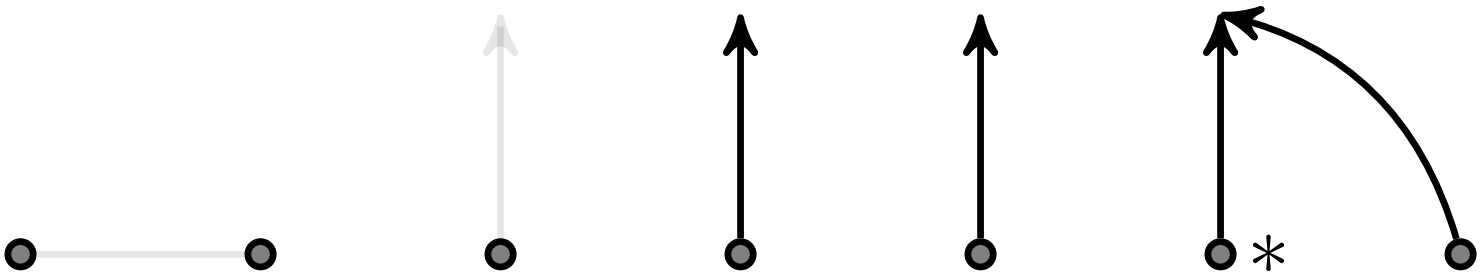

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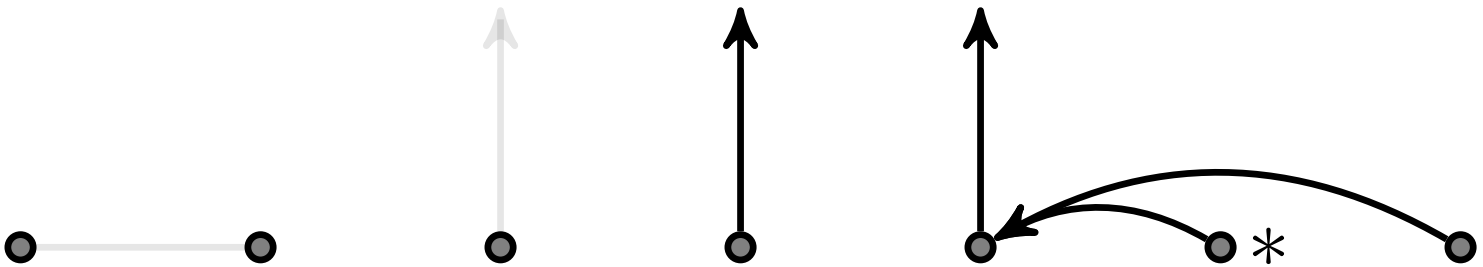

e_6 chooses $(e_5, 1)$, e_5 chooses $(e_4, 2)$.

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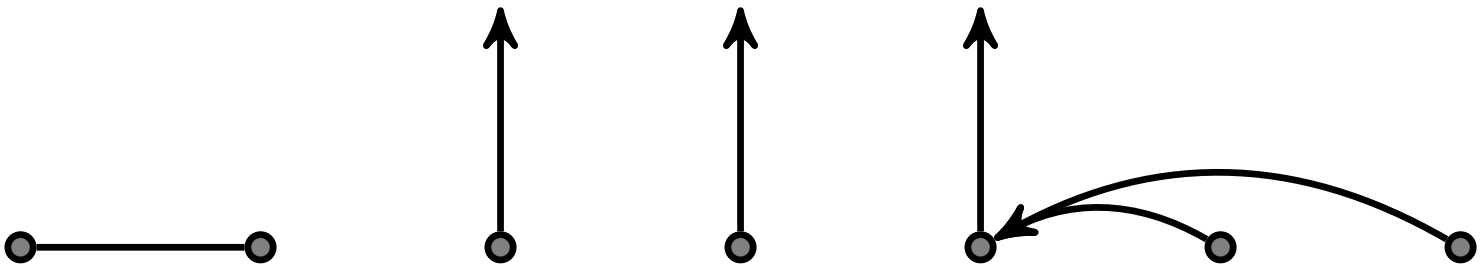

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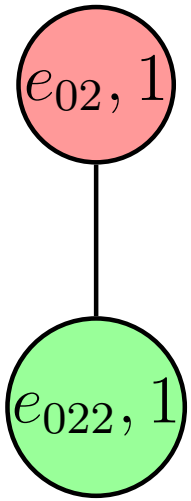

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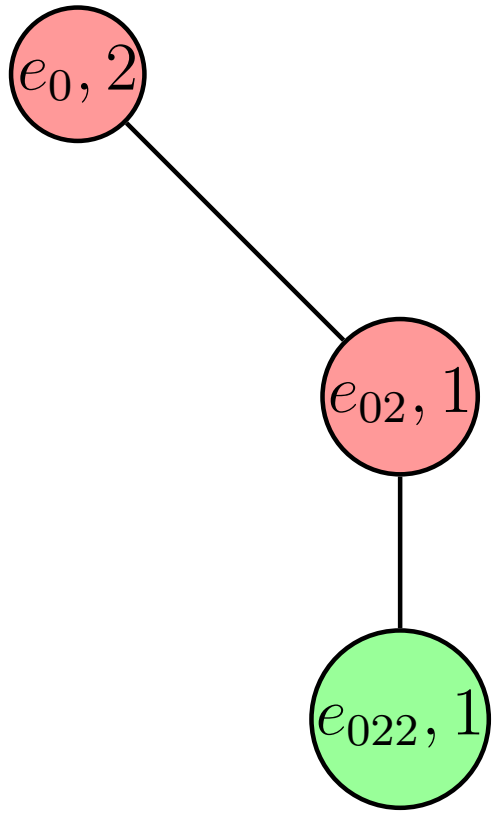

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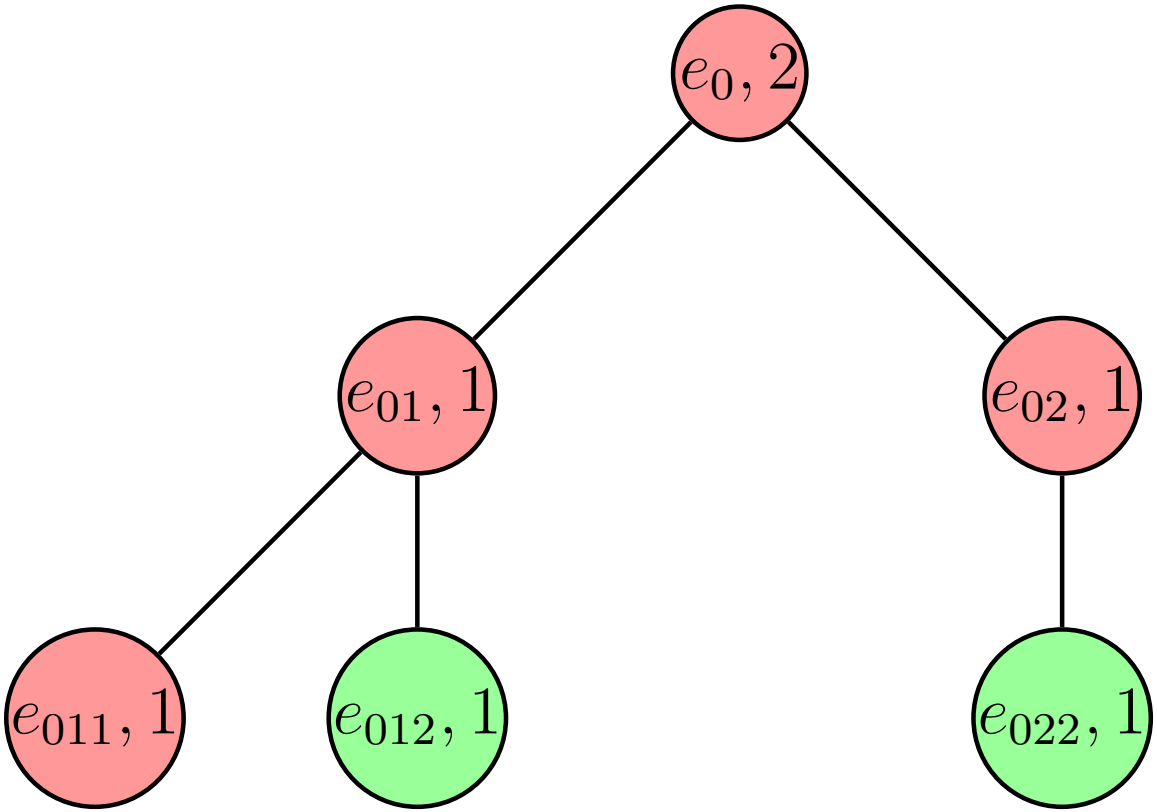

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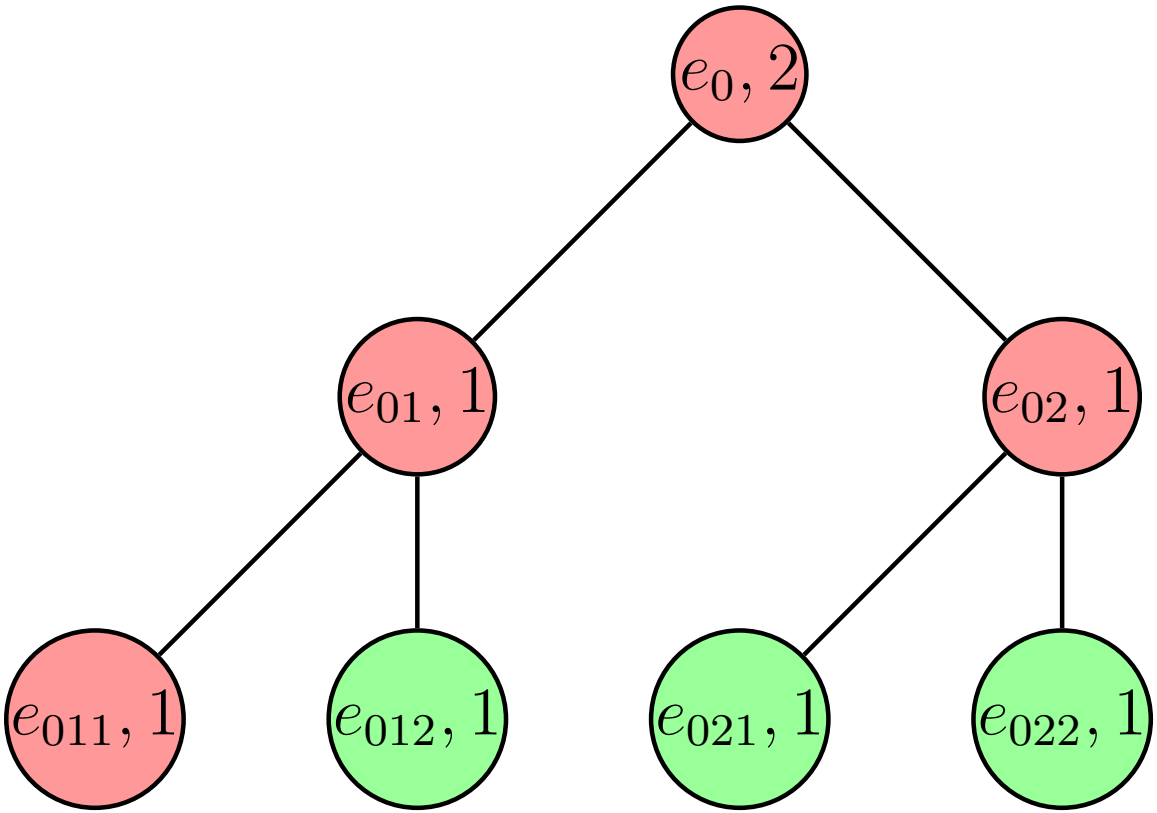

Future Questions

- Keep exploring this graph. What is the diameter?
- What is the exact size of the giant component? Tree sizes depend on the age of the root edge, making the problem difficult.
- Consider deleting the oldest vertex rather than the oldest edges.
- Uniform attachment graphs: new vertices choose old vertices uniformly at random.

Thank You!

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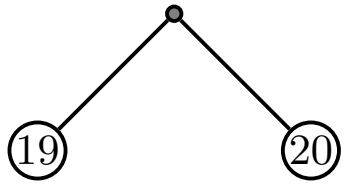
Giant Component

In Γ, this algorithm looks like this.

$S = \{19, 20\}$ 1_n v ν_n



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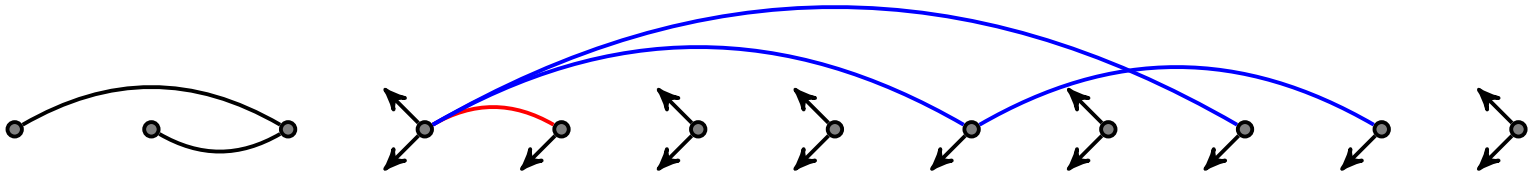

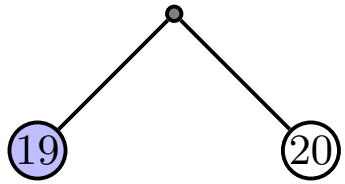

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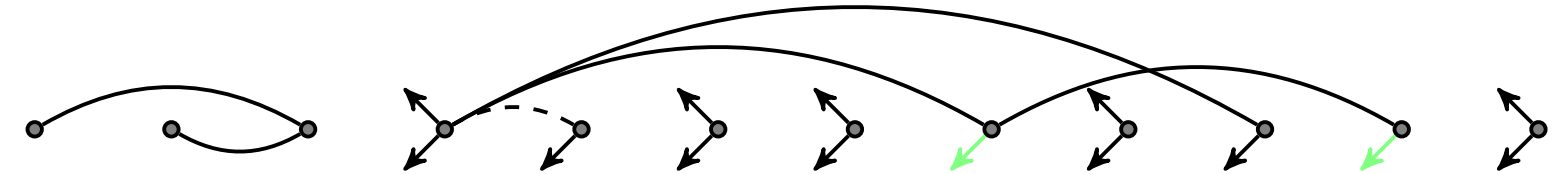

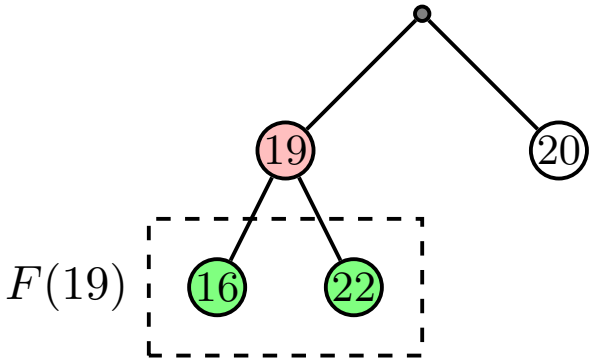

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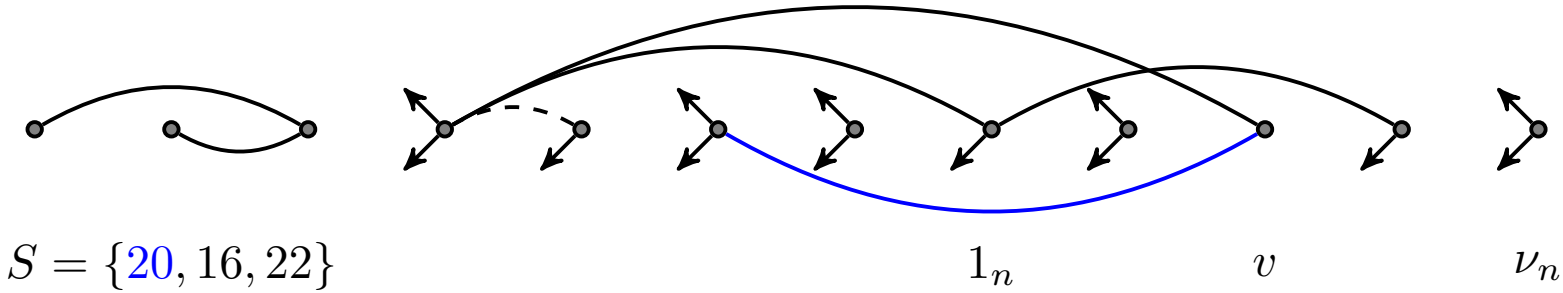

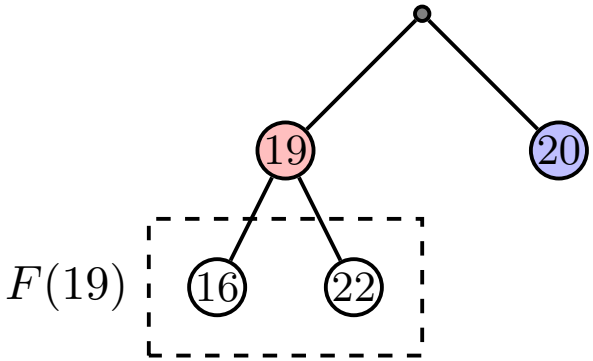

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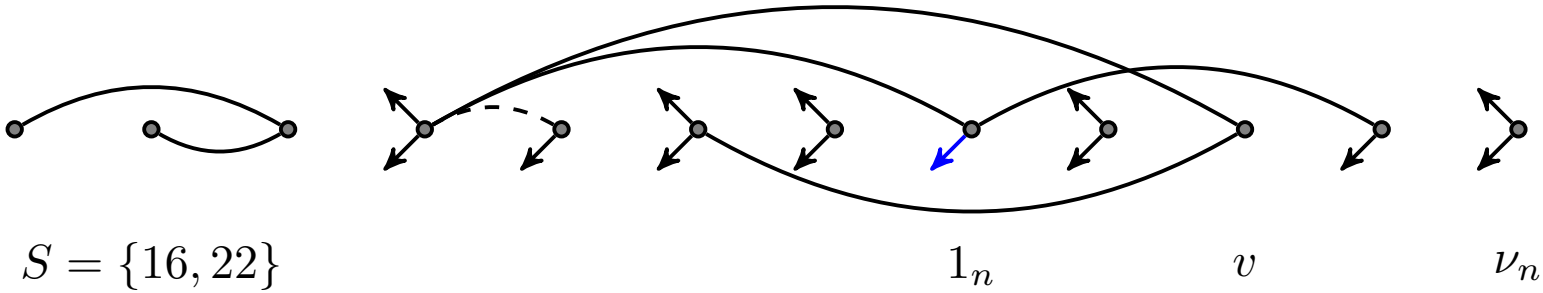

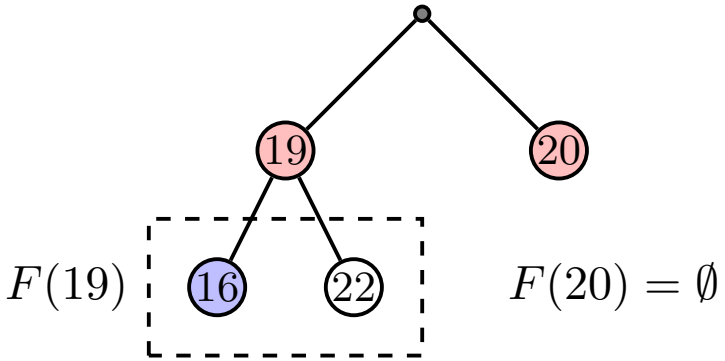

Giant Component

The size of the random tree can be bounded below by a Galton-Watson branching process.

Lemma

Let $m > 1$ and suppose $O(n^{1/2+\varepsilon})$ edges have been exposed. For any edge e, the size of $F(e)$ is stochastically bounded below by a random variable Z with distribution

$$Pr\{Z = k\} = \begin{cases} 0.26, & k = 0, \\ 0.46, & k = 1, \\ 0.28, & k = 2. \end{cases}$$

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Since $\mathbf{E}[Z] > 1$, this implies that the tree reaches size $\Omega(n^{1/2+\varepsilon})$ with some constant probability.

Giant Component

Proof now follows a standard template.

1. Number of large vertices: There are $\Omega(n)$ vertices in components of size $\Omega(n^{1/2+\varepsilon})$ (call them large), since the branching process survives with constant probability.
2. Unique giant: Whp there are no two disjoint large components.
3. Small components: Whp no tree terminates with size in range $[C \log n, n^{1/2+\varepsilon}]$.

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This shows that there is one unique component of size $\Omega(n)$ while all other components have size $O(\log n)$.