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The cover time of a biased random walk on the random 3-regular graph

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Uppsala

Joint work with
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Plan

- Definitions:
  Random walks and random regular graphs
- Results
- Idea behind proof
- A trick
- Some details

Cover time of a random walk

- Fix a graph $G = (V, E)$
- Pick some starting vertex v_0
- At time t move from v_{t-1} to v_t,
  chosen uniformly at random from neighbours of v_{t-1}
- Define random variable

$$C(G; v_0) = \min\{t : \{v_0, v_1, \ldots, v_t\} = V\}$$

- The vertex cover time of G is

$$C_V(G) = \max_{v_0 \in V} \mathbb{E}\left(C(G; v_0)\right)$$

- Edge cover time $C_E(G)$ is expected time to cover all edges

Cover time of a random walk

Feige (95): for any simple connected G on n vertices,

$$(1 - o(1))n \log n \leq C_V(G) \leq \frac{4n^3}{27}.$$

Applications:

- Graph exploration
- Message dissemination
- Web crawl
- Network search with only local information

Other random walks

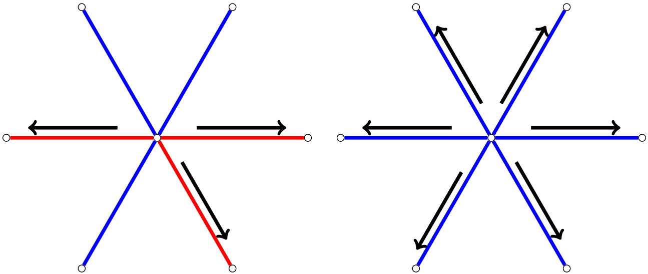
- Non-backtracking: avoid going from u to v to u
- Biased: choose vertices with probability based on degrees
- Biased/greedy: Avoid reusing edges

Slight name confusion!

- Today: biased random walk
- Introduced by Orenshtein and Shinkar (2014)
  as greedy random walk
- Introduced to speed up walk

Biased random walk

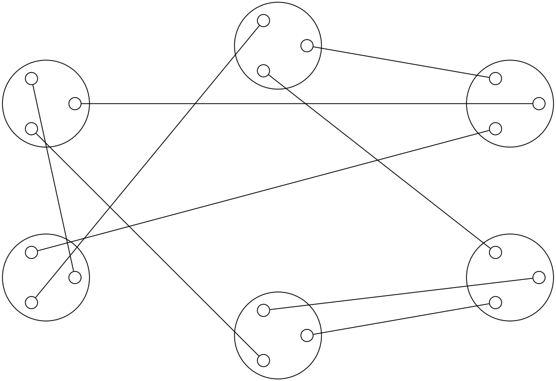
- Initially colour all edges red
- When edge is traversed, recolour it blue
- At time t, move from v_{t-1} along red edge chosen uniformly at random
  Only use blue edge if no red edges available



Random regular graph

Use configuration model

- Each vertex v associated with set $\mathcal{P}(v)$ of 3 *points*
- Pairing μ of $\bigcup_{v \in V} \mathcal{P}(v)$ chosen u.a.r.
- $u \in v$ if and only if $\mu(x) \in \mathcal{P}(v)$ for some $x \in \mathcal{P}(u)$



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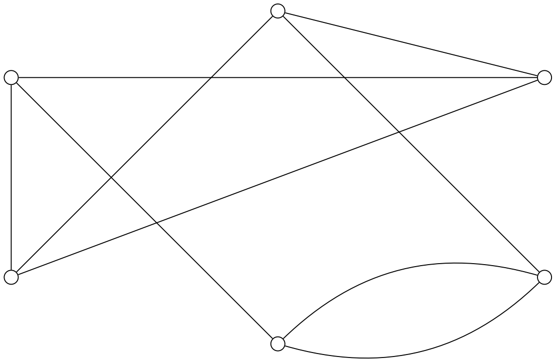
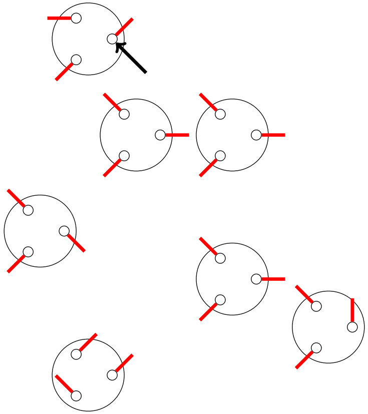


Walk on configuration model

Generate configuration as we go
Unused points chosen when available



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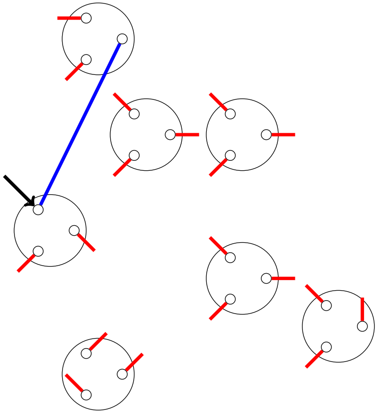

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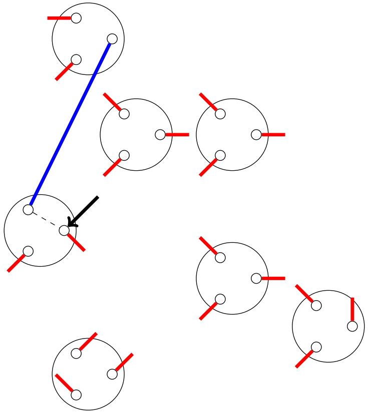

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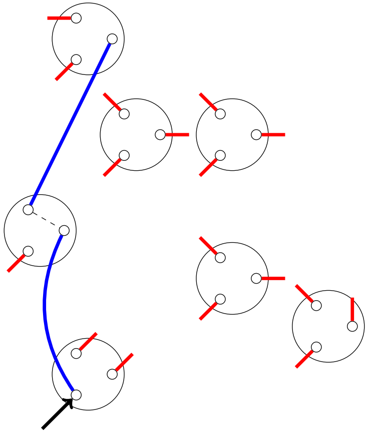

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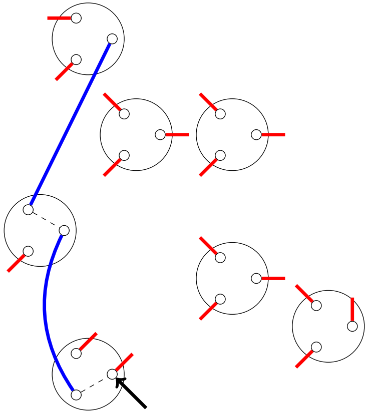

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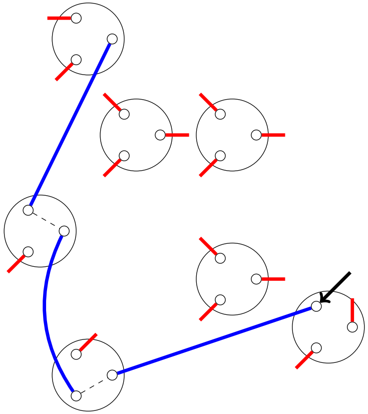

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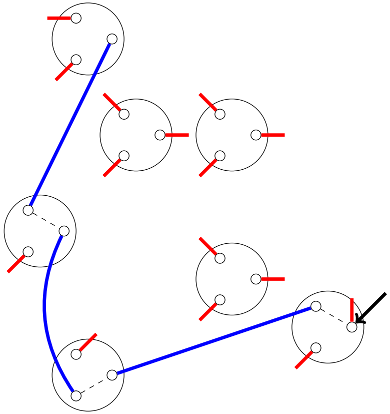

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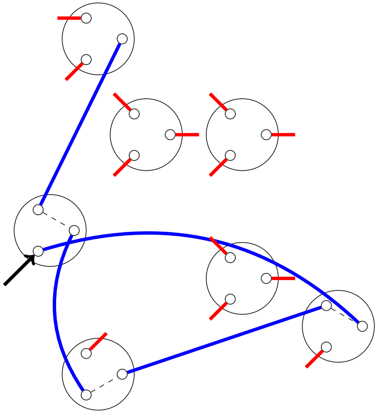

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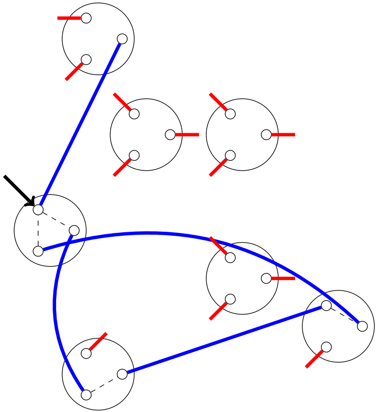

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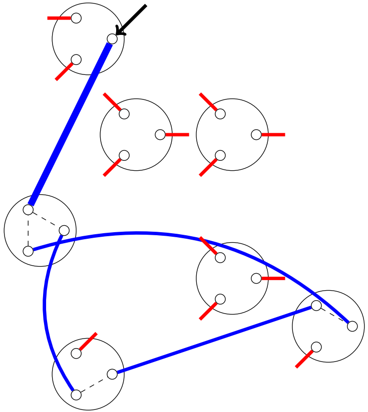

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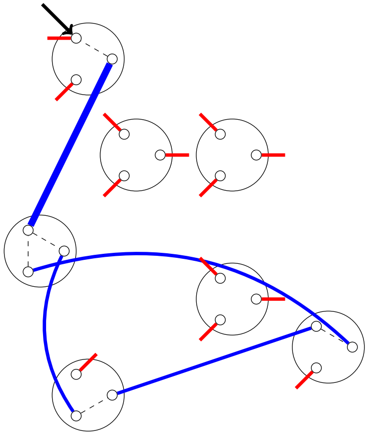

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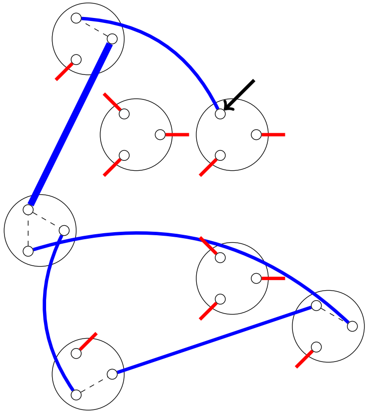

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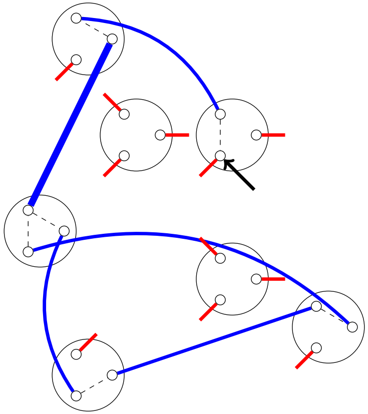

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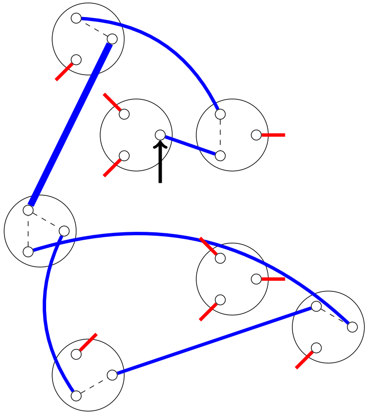

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Walks on random regular graphs

Let G_r denote the random r-regular graph, $r \geq 3$

Theorem (Cooper, Frieze 2005)

$$C_V^{simple}(G_r) \sim \frac{r-1}{r-2} n \log n.$$

Theorem (Cooper, Frieze 2016)

$$C_V^{non-backtracking}(G_r) \sim n \log n.$$

Theorem (Berenbrink, Cooper, Friedetzky 2015)

For $r \geq 4$ even,

$$C_V^{bias}(G_r) \sim \frac{rn}{2}.$$

Theorem (Cooper, Frieze, J.)

With high probability, G_3 is such that

$$C_V^{bias}(G_3) \sim n \log n.$$

Results

Theorem (Cooper, Frieze, J.)

With high probability, G_3 is such that

$$C_V^{bias}(G_3) \sim n \log n.$$

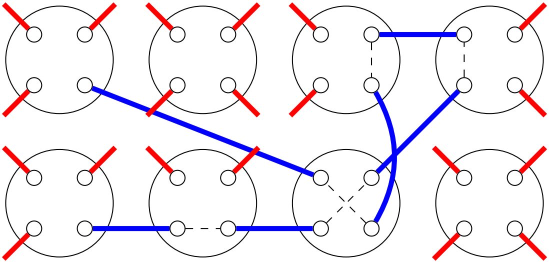
On arXiv:

Theorem (J.)

For $r \geq 5$ odd, w.h.p. G_r is such that

$$C_V^{bias}(G_r) \sim \frac{1}{r-2} n \log n.$$

Biased walk on 4-regular graph



The walk will almost always go from red edge to red edge,
as almost all vertices hold an even number of them

Results

Theorem

Define $C(t)$ as the number of steps used to find t distinct edges. W.h.p., G_3 is such that for all $3n/2(1 - o(1)) \leq t \leq 3n/2$,

$$C(t) \sim \frac{3}{2} n \log \left(\frac{3n}{3n - 2t + 1} \right)$$

Corollary

W.h.p., G_3 is such that

$$C_V^{bias}(G_3) \sim n \log n,$$

$$C_E^{bias}(G_3) \sim \frac{3}{2} n \log n.$$

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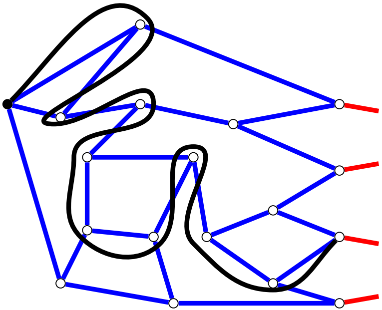
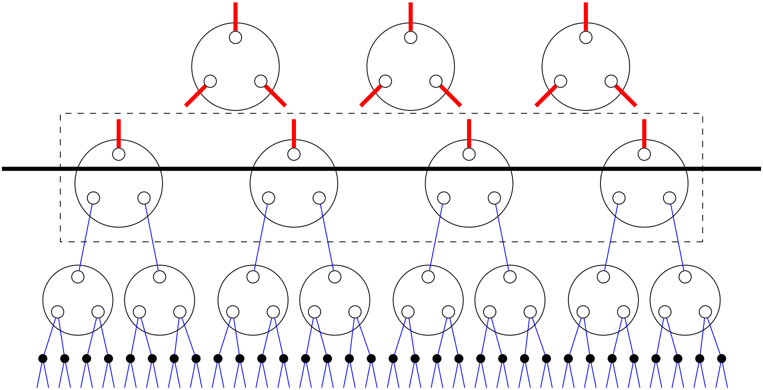
Proof idea

- Near end of process, almost all edges blue
- If walker surrounded by blue edges,
  locally behaves like simple random walk
- Use hitting time theory for simple random walks
  to estimate time to find red edge



Proof overview

What matters for hitting time is set of vertices with
exactly one red edge
("boundary")



Proof

Suppose

- G is 3-regular with positive spectral gap
- S is a vertex set such that
  - $G[S]$ is a perfect matching
  - no vertex of S is on a cycle of length $\leq \omega$
  - no two vertices of S are at S^c-distance $\leq \omega$

Then a simple random walk from a random starting point hits S in expected time $\approx 3n/|S|$

Proof

Suppose

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  - no two vertices of S are at S^c-distance $\leq \omega$

Then a simple random walk from a random starting point hits S in expected time $\approx 3n/|S|$

Also true if:

- $G[S]$ is almost a perfect matching
- almost no vertex of S is on a cycle of length $\leq \omega$
- almost no two vertices of S are at S^c-distance $\leq \omega$

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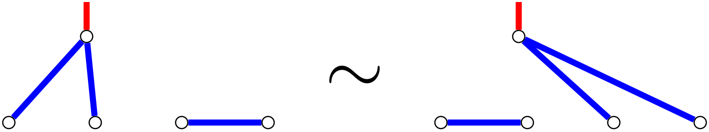
Rerandomizing boundary

Let $W(t)$ be graph induced on first t edges

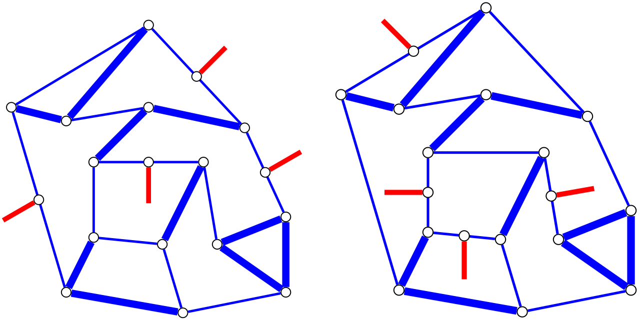
- Label edges $1, 2, \ldots, t$ in order of discovery
- Supposed edges $3, 4, 9$ are never revisited after discovery
- Also suppose $3, 4$ meet at vertex v
- v must be on boundary
- Replace $3, 4$ by one edge, detaching v
- Split 9 into two edges meeting at v

The resulting walk $W'(t)$ has $\Pr\{W'(t)\} = \Pr\{W(t)\}$



Rerandomizing boundary

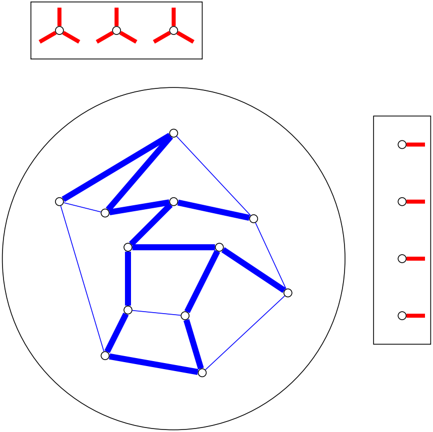
The two pictures below are equally likely...



...if the blue edges involved have been used exactly once

Sprinkling

Sprinkle boundary vertices “into” once-visited edges



Thick visited twice
Thin once

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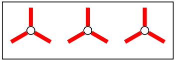

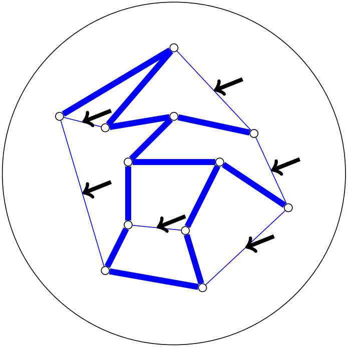

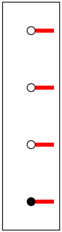

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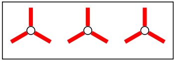

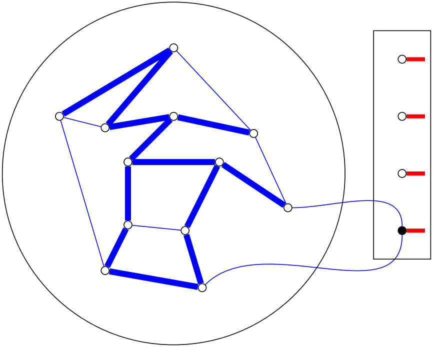

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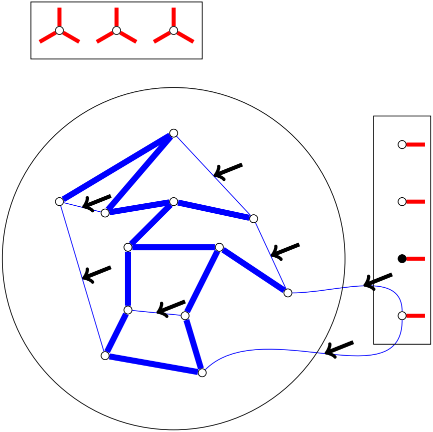

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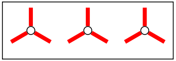

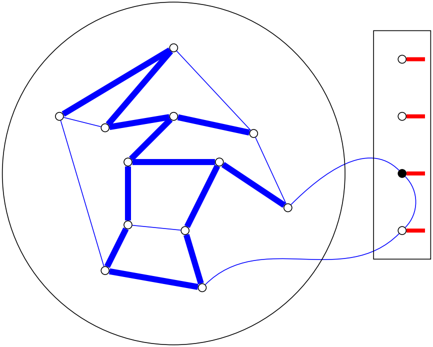

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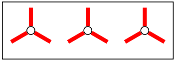

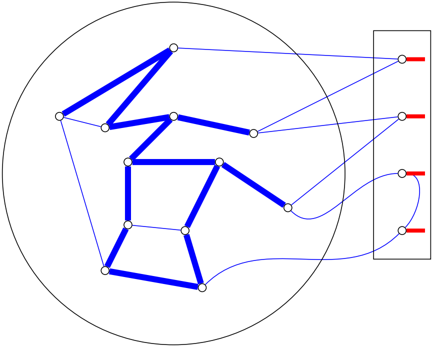

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A size lemma

Lemma

If $t = (1-\delta)\frac{3n}{2}$ *where* $\delta = o(1)$ *then*

$$\textit{number of unvisited vertices} = O\left(n\delta^{3/2}\right)$$

$$\textit{number of boundary vertices} \sim 3n\delta$$

$$\textit{number of once-visited edges} = \Omega\left(n\delta^{1/2}\right)$$

Conclusions:

- The $3n\delta/2$ red edges almost form a perfect matching
- Boundary vertices sprinkled into much larger set of once-visited edges, hence spread far apart

Set sizes: proof idea

- Hide boundary vertices
- Walk on blue edges
- Once-visited edge contains vertex w.p.

$$\frac{\#\text{boundary vertices}}{\#\text{once-visited edges}}$$

- Once vertex is found, walk to undiscovered vertex w.p.

$$3 \times \frac{\#\text{undiscovered vertices}}{\#\text{boundary vertices}},$$

  otherwise to a once-discovered vertex
- This sets up recursions for the three quantities

Set size recursions

Unvisited vertices:

$$\mathbb{E}\left(X_3(t+1)\right) \approx \left(1 - \frac{1}{3n-2t}\right) \mathbb{E}\left(X_3(t)\right),$$

Boundary vertices:

$$\mathbb{E}\left(X_1(t+1)\right) \approx \frac{3\mathbb{E}\left(X_3(t)\right)}{3n-2t} + \left(1 - \frac{2}{3n-2t}\right) \mathbb{E}\left(X_1(t)\right),$$

Once-visited edges:

$$\mathbb{E}\left(\Phi(t+1)\right) \approx 1 + \left(1 - \frac{1}{3n-2t}\right) \mathbb{E}\left(\Phi(t)\right)$$

Concluding remarks

- Unsurprisingly, the biased random walk is faster than the simple random walk in all known cases
- Open: Is there a G with

$$C_V^{bias}(G) > C_V^{simple}(G)?$$

- Open: Is the biased random walk recursive on $\mathbb{Z}^2$?

Thank you!