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Hamilton Cycles in Erdős–Rényi Subgraphs of Large Graphs

Probabilistic Midwinter meeting

Umeå Universitet

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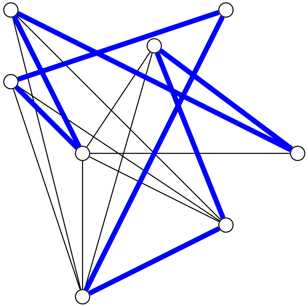
Hamilton cycles

- A Hamilton cycle is a cycle passing through each vertex exactly once
- A graph $G = (V, E)$ is Hamiltonian if it contains a Hamilton cycle



Hamilton cycles

- Minimum degree $\delta(G) \geq 2$ necessary for Hamilton cycle
- Minimum degree $\delta(G) \geq n/2$ sufficient for Hamilton cycle (Dirac '52)

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In random graph models the gap is usually much smaller

"Theorem"

Let G be a random graph from some well-known model. Then there should be some small k (usually 2 or 3) such that

G is Hamiltonian with high probability
if and only if
$\delta(G) \geq k$ with high probability.

The Erdős-Rényi process

- Let $\Gamma = (V, E)$ be some host graph (from a sequence (Γ_n))
- Suppose $|V| = n$ and $|E| = m$
- Order the edges $E = (e_1, \ldots, e_m)$ uniformly at random
- Define the Erdős-Rényi process (with host Γ) by

$$\Gamma_{n,t} = (V, \{e_1, e_2, \ldots, e_t\}), \quad 0 \leq t \leq m$$

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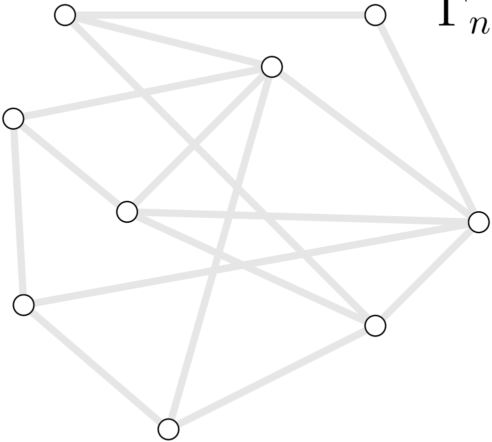
$$\Gamma_{n,t} = (V, \{e_1, e_2, \ldots, e_t\}), \quad 0 \leq t \leq m$$

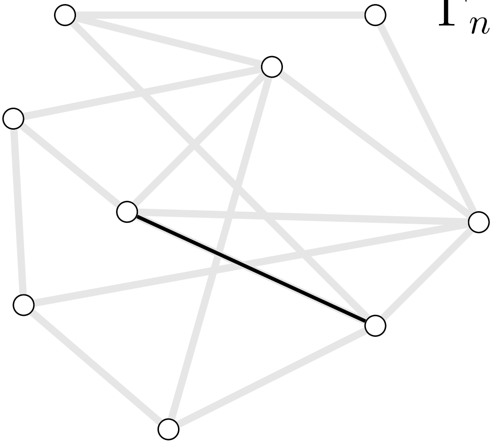
- Also define $\Gamma_{n,p}$ (the Erdős-Rényi subgraph) as the graph obtained by deleting any edge from Γ independently with probability $1 - p$
- $\Gamma_{n,t}$ and $\Gamma_{n,p}$ with $p = t/m$ are closely related

The Erdős-Rényi process



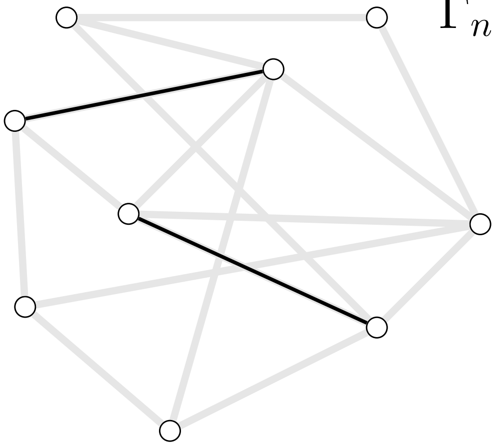
$\Gamma_{n,0}$

The Erdős-Rényi process



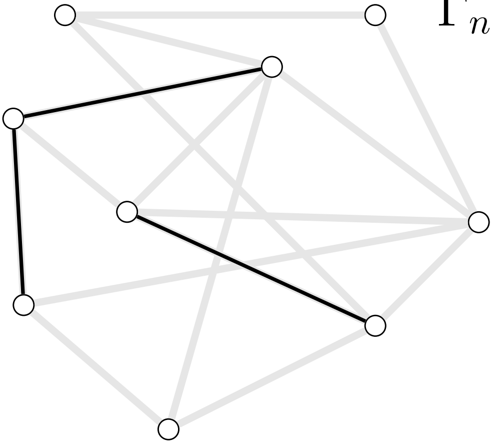
$\Gamma_{n,1}$

The Erdős-Rényi process



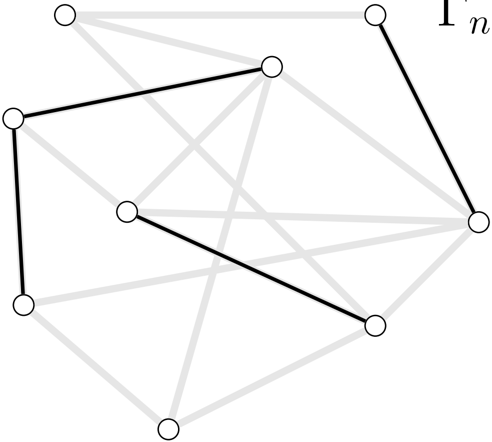
$\Gamma_{n,2}$

The Erdős-Rényi process



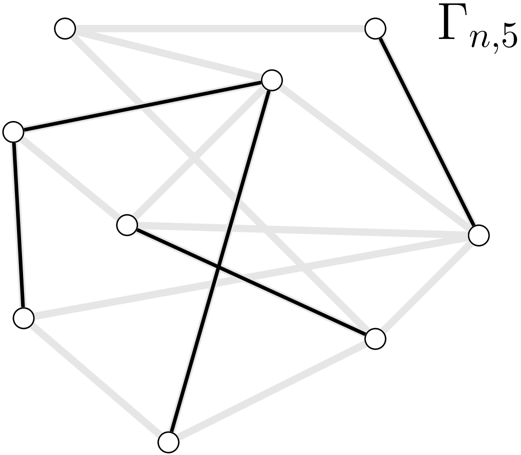
$\Gamma_{n,3}$

The Erdős-Rényi process

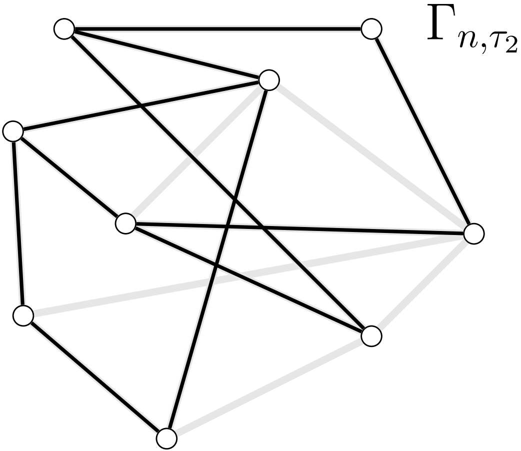


$\Gamma_{n,4}$

The Erdős-Rényi process



The Erdős-Rényi process



Hitting times

Given a Hamiltonian host Γ, define

$$\tau_2 = \min\{t : \delta(\Gamma_{n,t}) \geq 2\},$$

$$\tau_H = \min\{t : \Gamma_{n,t} \text{ Hamiltonian}\}.$$

Theorem (Ajtai-Komlós-Szemerédi 1985)

If $\Gamma = K_n$ then with high probability,

$$\tau_2 = \tau_H.$$

In other words, G_{n,τ_2} is Hamiltonian whp

The threshold

Theorem (Pósa '76, Koršunov '76, Komlós-Szemerédi '83)

Suppose

$$p = \frac{\log n + \log\log n + c_n}{n}.$$

Then

$$\lim_{n\to\infty} Pr\{G_{n,p}\ Hamiltonian\} = \begin{cases} 0, & c_n \to -\infty, \\ e^{-e^{-c}}, & c_n \to c, \\ 1, & c_n \to \infty. \end{cases}$$

Frieze ('85) found a similar result for $\Gamma = K_{n,n}$.

New results

Say (Γ_n) is a Strong Dirac Graph sequence (SDG)
if there exists some $\varepsilon > 0$ such that $\delta(\Gamma_n) \geq (1/2+\varepsilon)n$ for all large n.

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Theorem (J. 2019+)

Suppose (Γ_n) is an SDG. Then with high probability,

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The threshold for Hamiltonicity in $\Gamma_{n,p}$ is the unique solution p_0 to

$$\sum_{v\in\Gamma_n}(1-p)^{d_\Gamma(v)}\log n = 1. \tag{1}$$

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$$\sum_{v \in \Gamma_n} (1-p)^{d_\Gamma(v)} \log n = 1. \tag{1}$$

LHS in (1) $\approx$ expected number of vertices of degree less than 2.

Regular host graphs

Corollary

Suppose Γ_n is βn-regular for some $\beta > 1/2$ (for all n). Let

$$p = \frac{\log n + \log\log n + c_n}{\beta n}.$$

Then

$$\lim_{n\to\infty} Pr\left\{\Gamma_{n,p} \text{ } Hamiltonian\right\} = \begin{cases} 0, & c_n \to -\infty, \\ e^{-e^{-c}}, & c_n \to c, \\ 1, & c_n \to \infty. \end{cases}$$

"It is very pleasing, though perhaps not surprising, that a result similar to [P, K, KS] can be proved" – Frieze, '85

What we will show

Theorem

Suppose $\omega \to \infty$ *arbitrarily slowly with n and let*

$$t = \tau_2 + \omega n.$$

Then

$$\lim_{n\to\infty} Pr\{\Gamma_{n,t} \text{ } \mathit{Hamiltonian}\} = 1.$$

Proof plan

Will show: $\Gamma_{n,t}$ Hamiltonian whp for $t = \tau_2 + \omega n$

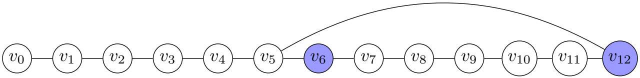
- Pósa's rotation lemma
- Pósa extensions
- Proof for $\Gamma = K_n$
- Rotation-extension with smaller Γ
- A colouring argument
- A more refined colouring argument

Pósa's lemma

- Let G be a non-Hamiltonian graph
- Suppose $P = (v_0, v_1, \ldots, v_\ell)$ is a longest path in G
- If $v_\ell \sim v_i$, $i < \ell - 1$, we obtain the path

$$P' = (v_0, v_1, \ldots, v_i, v_\ell, v_{\ell-1}, \ldots, v_{i+1})$$

  via a rotation with v_0 fixed
- P' is also a longest path
- Let $\mathrm{EP}(v_0)$ be the set of endpoints obtainable with v_0 fixed (so $v_\ell, v_{i+1} \in \mathrm{EP}(v_0)$)



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Lemma (Pósa '76)

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Definition

A graph G expands if there is some $\alpha > 0$ such that

$$|S| \leq \alpha n \quad \text{implies} \quad |N(S)| \geq 2|S|.$$

Corollary

In a non-Hamiltonian expander, $|N(\mathrm{EP}(v_0))| \geq \alpha n$*.*

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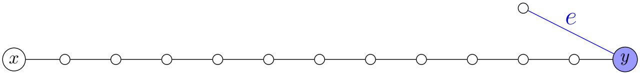

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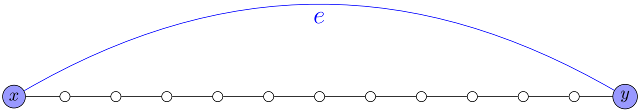
  2. If e connects some $x \in \mathrm{EP}$ to $y \in \mathrm{EP}(x)$: cycle of length $\ell + 1$
     - If $\ell = n - 1$, this is a Hamilton cycle.



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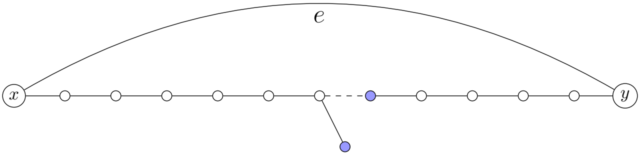
     - Otherwise, as G is connected, there exists a path of length $> \ell$.



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    - If $\ell = n - 1$, this is a Hamilton cycle.
    - Otherwise, as G is connected, there exists a path of length $> \ell$.
3. In other cases, e is not useful.

Say that e is an extender in cases 1 and 2.

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Proof for $\Gamma = K_n$

- Easy: $G_{n,t}$ is a connected expander whp for $t = \frac{1}{2}(n \log n + n \log \log n + \omega n)$
- If $G_{n,t}$ not already Hamiltonian, let x be an endpoint
- Consider the edge set

$$F = \bigcup_{y \in \mathrm{EP}(x)} \{y\} \times \mathrm{EP}(y).$$

We have at least $|F| \geq \left(\frac{\alpha n}{2}\right)^2$ extenders

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- Whp, adding ωn edges gives n extenders, so

$$\tau_H \leq t + \omega n$$

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Smaller host graphs

- Easy: $\Gamma_{n,t}$ is a connected expander whp for $t = p_0 m + \omega n$
- If $\Gamma_{n,t}$ not already Hamiltonian, let x be an endpoint
- Consider the edge set

$$F = E(\Gamma) \cap \bigcup_{y \in \mathrm{EP}(x)} \{y\} \times \mathrm{EP}(y).$$

We might have $|F| = 0$

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The issue: we can only add edges from the host Γ,
and it may be that no extenders are in Γ

Example: $\Gamma = K_{n,n}$ and the longest path has odd length

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Smaller host graphs

- Let $G \subseteq \Gamma$ be a connected, non-Hamiltonian expander
- Suppose G has $o(n^2)$ extenders with respect to Γ
- Recall $|\mathrm{EP}(x)| \geq \alpha n$ for all $x \in \mathrm{EP}$
- Say x is bad if it has $o(n)$ edges to $\mathrm{EP}(x)$ (in Γ)

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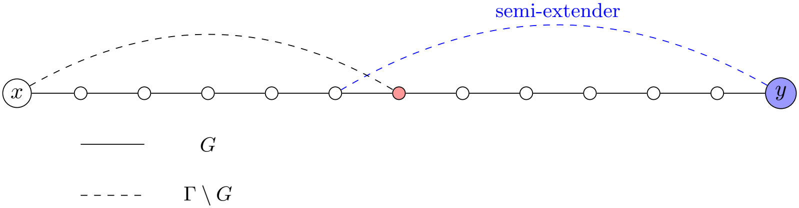
- If x is bad and $y \in \text{EP}(x)$, this picture happens $\geq \varepsilon n$ times:



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- With probability $1 - ne^{-\Omega(\omega n)} = 1 - e^{-\Omega(\omega n)}$, all $x \in \mathrm{EP}$ are now good

Creating extenders

- We started with a connected non-Hamiltonian expander $G \subseteq \Gamma$
- We added ωn random edges from Γ to G
- We now have $\Omega(n^2)$ extenders wrt Γ in G
- Adding another ω edges will extend the path

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Issue: This technique might require adding ωn^2 edges (impossible)

The argument will turn out to be useful anyway

Smaller host graphs

Let's summarize the last few slides in a lemma.
We let $\ell(G)$ denote the length of the longest path in G, with $\ell(G) = n$ if G is Hamiltonian.

Lemma ("Extension lemma")

Suppose $G \subseteq \Gamma$ is a connected non-Hamiltonian expander. Let G^+ be the random graph obtained by adding ωn random edges from Γ to G. Then

$$Pr\left\{\ell(G^+) > \ell(G)\right\} = 1 - e^{-\Omega(\omega n)}$$

Proof plan

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- The probabilities don't match, and we can use this

A colouring argument

- We will define a random subgraph $\Gamma^*_{n,t} \subseteq \Gamma_{n,t}$
- Define the event

$$\mathcal{A} = \{\ell(\Gamma^*_{n,t}) = \ell(\Gamma_{n,t}) < n\},$$

i.e. "neither graph is Hamiltonian and they share a longest path"

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- Then

$$\Pr\{\mathcal{P}_k\} = \frac{\Pr\{\mathcal{A} \cap \mathcal{P}_k\}}{\Pr\{\mathcal{A} \mid \mathcal{P}_k\}}$$

Choosing $\Gamma^*_{n,t}$

- Let $q = \omega / \log n$
- Independently colour each edge in $\Gamma_{n,t}$ red with probability q, and blue otherwise
- Let $\Gamma^*_{n,t}$ be the blue graph, i.e. remove all red edges

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  (this is the probability that any fixed path of length k is all-blue)
- So for any event of the form $\mathcal{E} = \{\Gamma_{n,t} \in \mathcal{G}\}$,

$$\Pr\{\mathcal{A} \mid \mathcal{E}\} \geq e^{-\Theta(\omega n/\log n)}$$

The main calculation

With such a choice of $\Gamma^*_{n,t}$ we have

$$\Pr\{\mathcal{P}_k\} = \frac{\Pr\{\mathcal{A} \cap \mathcal{P}_k\}}{\Pr\{\mathcal{A} \mid \mathcal{P}_k\}} \leq \Pr\{\mathcal{A} \cap \mathcal{P}_k\} e^{\Theta(\omega n / \log n)}$$

The main calculation

With such a choice of $\Gamma^*_{n,t}$ we have

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- Aim: use the lemma to show that

$$\Pr\{\mathcal{A} \cap \mathcal{P}_k\} \leq e^{-\Omega(\omega n)}$$

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Define

$$\mathcal{P}_k^* = \{\ell(\Gamma_{n,t}^*) = k\}.$$

We have

$$\Pr\{\mathcal{A} \cap \mathcal{P}_k\} = \Pr\{\mathcal{P}_k^* \cap \mathcal{P}_k\} = \Pr\{\mathcal{P}_k \mid \mathcal{P}_k^*\}\Pr\{\mathcal{P}_k^*\} \leq \Pr\{\mathcal{A} \mid \mathcal{P}_k^*\}$$

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Also need

$$\mathcal{C}^* = \{\Gamma_{n,t}^* \text{ is connected}\}$$

$$\mathcal{E}^* = \{\Gamma_{n,t}^* \text{ expands}\}$$

Proof plan

Will show: $\Gamma_{n,t}$ Hamiltonian whp for $t = \tau_2 + \omega n$

- Pósa's rotation lemma
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Bringing in expansion

More refined approach: for any events $\mathcal{M}, \mathcal{N}$,

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- We will use

$$\Pr\{\mathcal{A} \cap \mathcal{M} \mid \mathcal{P}_k \cap \mathcal{N}\} \geq \Pr\{\mathcal{A} \mid \mathcal{P}_k \cap \mathcal{N}\} - \Pr\left\{\overline{\mathcal{M}} \mid \mathcal{P}_k \cap \mathcal{N}\right\},$$

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What we need for the numerator:

- Let $\mathcal{B}_k = (\mathcal{M} \cap \mathcal{N} \cap \mathcal{P}_k) \setminus (\mathcal{C}^* \cap \mathcal{E}^*)$. Then

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- First term, using the extension lemma, is bounded by

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Constraint roundup

We also need $\Pr\{\mathcal{N}\} = 1 - o(1)$ so that

$$\Pr\{\Gamma_{n,t} \text{ not Hamiltonian}\} \leq \Pr\{\overline{\mathcal{N}}\} + \sum_k \Pr\{\mathcal{P}_k \cap \mathcal{N}\} = o(1).$$

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Pick events $\mathcal{M}, \mathcal{N}$ such that

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- $\Pr\{\mathcal{N}\} = 1 - o(1)$,
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The underlying challenge

- For t in our range, we only have

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- We will redefine $\Gamma^*_{n,t}$ so that it expands when $\Gamma_{n,t}$ does

Redefine $\Gamma^*_{n,t}$

- Colour edges in $\Gamma_{n,t}$ red with probability $q = \frac{\omega}{\log n}$, else blue

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- $\Gamma^*_{n,t}$ and $\Gamma_{n,t}$ agree on properties only concerning SMALL and edges incident to them

Lemma

We have

$$\mathcal{D} \cap \mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \{\Gamma_{n,t} \text{ expands}\}$$

where

$$\mathcal{D} = \{\delta(\Gamma_{n,t}) \geq 2\},$$

$$\mathcal{E}_1 = \{\text{every vertex set } |S| \leq 6\alpha n \text{ has } e(S) \leq \frac{\log n}{1000}|S| \text{ in } \Gamma_{n,t}\},$$

$$\mathcal{E}_2 = \{\Gamma_{n,t} \text{ contains no paths or cycles of length} \leq 4 \text{ involving two edges incident to SMALL}\}.$$

$(\alpha = e^{-2000})$

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The point: $\mathcal{D}, \mathcal{E}_1, \mathcal{E}_2$ immediately transfer to $\Gamma^*_{n,t}$

Picking $\mathcal{M}, \mathcal{N}$

We let

$$\mathcal{N} = \{\delta(\Gamma_{n,t}) \geq 2\} \cap \mathcal{E}_1 \cap \mathcal{E}_2$$

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What we needed was:

- $\mathcal{N}$ concerns only $\Gamma_{n,t}$ (by design)
- $\Pr\{\mathcal{N}\} = 1 - o(1)$ (routine bounds)
- $\Pr\left\{\overline{\mathcal{M}} \mid \mathcal{P}_k \cap \mathcal{N}\right\} \leq e^{-\Omega(\omega n/\log n)}$
  (expansion makes connectivity likely)
- $\Pr\{(\mathcal{M} \cap \mathcal{N} \cap \mathcal{P}_k) \setminus (\mathcal{C}^* \cap \mathcal{E}^*)\} \leq e^{-\Omega(\omega n)}$
  (is actually zero since $\mathcal{M} = \mathcal{C}^*, \mathcal{N} \subseteq \mathcal{E}^*$)

We are done

We have

$$\Pr\left\{\Gamma_{n,t} \text{ not Hamiltonian}\right\} \le \Pr\left\{\overline{\mathcal{N}}\right\} + \sum_{k<n} \Pr\left\{\mathcal{P}_k \cap \mathcal{N}\right\}$$

and

$$\begin{aligned}\Pr\left\{\mathcal{P}_k \cap \mathcal{N}\right\} &= \frac{\Pr\left\{\mathcal{A} \cap \mathcal{M} \cap \mathcal{P}_k \cap \mathcal{N}\right\}}{\Pr\left\{\mathcal{A} \cap \mathcal{M} \mid \mathcal{P}_k \cap \mathcal{N}\right\}} \\ &\le \frac{e^{-\Omega(\omega n)}}{e^{-\Theta(\omega n/\log n)}} \\ &= o(n^{-1}).\end{aligned}$$

Proof plan

Will show: $\Gamma_{n,t}$ Hamiltonian whp for $t = \tau_2 + \omega n$

- Pósa's rotation lemma
  - Non-Hamiltonian expanders have $|\mathrm{EP}(x)| \geq \alpha n$
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  - Need $\Gamma^*_{n,t}$ to be a connected expander
- A more refined colouring argument
  - Make $\Gamma^*_{n,t}$, $\Gamma_{n,t}$ agree on low-degree vertices
  - Make expansion transfer from $\Gamma_{n,t}$ to $\Gamma^*_{n,t}$

Summary

Why does this work?

- Most edges are between vertices of high degree ($\Omega(\log n)$)
- Such edges are unlikely to be in longest path
- Randomly removing ωn such edges is fairly likely to be safe (longest path remains and the graph still expands)
- Randomly throwing them back in is likely to extend path
- This "contradiction" is formalized by

$$\Pr\{\mathcal{P}_k \cap \mathcal{N}\} = \frac{\Pr\{\mathcal{A} \cap \mathcal{M} \cap \mathcal{P}_k \cap \mathcal{N}\}}{\Pr\{\mathcal{A} \cap \mathcal{M} \mid \mathcal{P}_k \cap \mathcal{N}\}}$$

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- This argument seems suited for spanning structures in random subgraphs of non-complete host graphs

Summary

What were the keys?

- We redefined $\Gamma^*_{n,t}$ so that it agrees with $\Gamma_{n,t}$ on vertices of low degree
- We considered a stricter type of expansion which automatically transfers from $\Gamma_{n,t}$ to $\Gamma^*_{n,t}$

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Things I skipped:

- How expansion follows from decreasing events and $\delta \geq 2$
- We can consider Γ_{n,τ_2} without too much trouble, but it adds even more events to the calculations
- We should have conditioned on $\Omega(\omega n)$ red edges being removed
- When we re-place the red edges, they must be in LARGE. Should have argued that $|\text{LARGE}| = n - o(n)$ (more events)